§14.5 The Chain Rule

Let us first recall the chain rule for functions of a single variable.

Given $y = f(x)$ and $z = g(y)$, the derivative of the composition $z = g \circ f(x) = g(f(x))$ is

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

or

$$[g \circ f(x)]' = g'(f(x))f'(x)$$

Diagram:

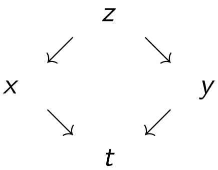
$$z \xrightarrow{d} y \xrightarrow{d} x$$

The Chain Rule (Case 1)

Suppose $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, and assume that all functions are differentiable. Then z is a differentiable function of t given by the composition $z = f(g(t), h(t))$ with derivative

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Tree Diagram:



Example

If $z = x^2y + 3xy^4$*, where* $x = \sin(2t)$ *and* $y = \cos t$*, find* $\dfrac{dz}{dt}$ *when* $t = 0$*.*

Solution.

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2xy + 3y^4)\cos(2t) \cdot 2 + (x^2 + 3x \cdot 4y^3)(-\sin t) \\ &= 2(2xy + 3y^4)\cos(2t) - (x^2 + 12xy^3)\sin t \end{aligned}$$

Example

If $z = x^2y + 3xy^4$, *where* $x = \sin(2t)$ *and* $y = \cos t$, *find* $\frac{dz}{dt}$ *when* $t = 0$.

Solution.

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2xy + 3y^4)\cos(2t) \cdot 2 + (x^2 + 3x \cdot 4y^3)(-\sin t) \\ &= 2(2xy + 3y^4)\cos(2t) - (x^2 + 12xy^3)\sin t \end{aligned}$$

When $t = 0$, $x = \sin(2 \cdot 0) = 0$ and $y = \cos 0 = 1$, so

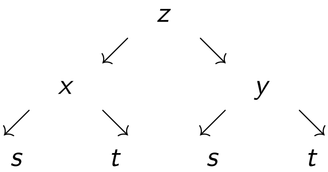
$$\frac{dz}{dt}|_{t=0} = 2(2 \cdot 0 \cdot 1 + 3 \cdot 1^4)\cos(2 \cdot 0) - (0^2 + 12 \cdot 0 \cdot 1^3)\sin 0 = 6.$$

The Chain Rule (Case 2)

Suppose $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, and assume that all functions are differentiable. Then z is a differentiable function of (x, t) given by the composition $z = f(g(s, t), h(s, t))$ with partial derivatives

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Tree Diagram:



Example

If $z = e^x \sin y$, *where* $x = st^2$ *and* $y = s^2 t$, *find* $\dfrac{\partial z}{\partial s}$.

Solution.

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= (e^x \sin y)t^2 + (e^x \cos y)(2st) \\
&= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t)
\end{aligned}$$

The Chain Rule (General Version)

Suppose $u = u(x_1, x_2, \ldots, x_n)$ and $x_i = x_i(t_1, t_2, \ldots, t_m)$, $i = 1, \ldots, n$, and assume that all functions are differentiable. Then

$$\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}.$$

for $j = 1, \ldots, m$.

Example

If $u = x^4y + y^2z^3$, *where* $x = rse^t$, $y = rs^2e^{-t}$ *and* $z = r^2 s \sin t$, *find* $\frac{\partial u}{\partial s}$ *when* $(r, s, t) = (2, 1, 0)$.

Solution.

$$\begin{aligned} \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} \\ &= 4x^3yre^t + (x^4 + 2yz^3)(2rse^{-t}) + 3y^2z^2r^2 \sin t \end{aligned}$$

When $(r, s, t) = (2, 1, 0)$, we have $(x, y, z) = (2, 2, 0)$, then

$$\frac{\partial u}{\partial s}\Big|_{(r,s,t)=(2,1,0)} = 64 \cdot 2 + 16 \cdot 4 + 0 \cdot 0 = 192.$$

Example

Find the derivative of $f(x, y, z) = x^2 + y^3 + \sin z$ *along the helix* $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$.

Solution.

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= 2x(-\sin t) + 3y^2 \cos t + \cos z \cdot 1 \\ &= -2\cos t \sin t + 3\cos t \sin^2 t + \cos t \end{aligned}$$

Implicit Differentiation

Assume that the equation $F(x, y, z) = 0$ defines z implicitly as a function of (x, y), we can use chain rule to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Indeed, take $\frac{\partial}{\partial x}$ on both sides of $F(x, y, z) = 0$, then

$$F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x} = 0,$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, get

$$F_x + F_z \frac{\partial z}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.$$

Similarly, by taking $\frac{\partial}{\partial y}$, we have

$$F_y + F_z \frac{\partial z}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Example

Find $\dfrac{\partial z}{\partial x}$ *if* $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution. Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$, then

$$F_x = 3x^2 + 6yz, \qquad F_z = 3z^2 + 6xy$$

Thus,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$