§12.5 Equations of Lines and Planes

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Equations of Planes
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Distance From A Point to A Plane

Vector Equation of Lines

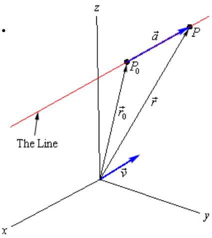
Let L be a line through the point P_0 (reference point) and parallel to the vector $\mathbf{v}$ (direction vector). Take an arbitrary point P on L, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. The **vector equation** of the line L is given by

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Proof.

$\mathbf{a} = \overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = \mathbf{r} - \mathbf{r}_0.$

Since
$\mathbf{a}$ is parallel to $\mathbf{v}$,
we have $\mathbf{a} = t\mathbf{v}$
for some $t \in \mathbb{R}$.



Parametric Equation of Lines

In $\mathbb{R}^2$, if the points $P_0 = (x_0, y_0)$, $P = (x, y)$, and the direction vector $\mathbf{v} = \langle a, b \rangle$, we rewrite the vector equation as

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t\langle a, b \rangle = \langle x_0 + ta, y_0 + tb \rangle$$

or **Parametric Equations:** $$\begin{cases} x = x_0 + ta, \\ y = y_0 + tb. \end{cases}$$

In $\mathbb{R}^3$, if the points $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$, and the direction vector $\mathbf{v} = \langle a, b, c \rangle$, we rewrite the vector equation as

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

or **Parametric Equations:** $$\begin{cases} x = x_0 + ta, \\ y = y_0 + tb, \\ z = z_0 + tc. \end{cases}$$

Symmetric Equation of Lines

In the parametric equation $\begin{cases} x = x_0 + ta, \\ y = y_0 + tb, \\ z = z_0 + tc, \end{cases}$ if none of **direction numbers** a, b, c is 0, we rewrite (to get rid of the parameter t)

Symmetric Equation: $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

In case when one of a, b, c is 0, for example, $a = 0$ but $b, c \neq 0$, the symmetric equations of the line is then given by

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Similarly, we can write down symmetric equations for a line in $\mathbb{R}^2$.

Example

Find vector equation, parametric equations and symmetric equation for the line containing the points $P = (1, 2, -3)$ *and* $Q = (3, -2, 1)$.

Solution: The reference point $P = (1, 2, -3)$ and direction vector

$$\mathbf{v} = \overrightarrow{PQ} = \langle 3 - 1, -2 - 2, 1 - (-3) \rangle = \langle 2, -4, 4 \rangle.$$

$$\textbf{Vector Equation: } \mathbf{r} = \langle 1, 2, -3 \rangle + t\langle 2, -4, 4 \rangle \ ;$$

$$\textbf{Parametric Equation: } \begin{cases} x = 1 + 2t, \\ y = 2 - 4t, \\ z = -3 + 4t. \end{cases}$$

$$\textbf{Symmetric Equation: } \frac{x-1}{2} = \frac{y-2}{-4} = \frac{z-(-3)}{4}.$$

Example

Let L_1 be the line given by the symmetric equation

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{1}$$

1. *At what points does Line L_1 intersect xy-plane?*
2. *Find Line L_2 through $Q(1,3,-2)$ and parallel to L_1.*
3. *Find Line L_3 through $Q(1,3,-2)$ perpendicular to and intersecting L_1.*

Solution:

1. If L_1 intersects xy-plane at $P=(x,y,z)$, then $z=0$, thus,

   $$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{0-3}{1},$$

   so $x=-5$ and $y=5$. Therefore, L_1 intersects xy-plane at $(-5,5,0)$.

2. The direction vector is $\mathbf{v} = \langle 2, -1, 1 \rangle$ gives direction of L_2. Since L_2 passes through $Q(1, 3, -2)$, we have

$$\text{Vector Equation for Line } L_2: \ \mathbf{r} = \langle 1, 3, -2 \rangle + t\langle 2, -1, 1 \rangle.$$

3. The parametric equation of L_1 is $\begin{cases} x = 1 + 2t, \\ y = 2 - t, \\ z = 3 + t. \end{cases}$ Let $R = (1 + 2t_0, 2 - t_0, 3 + t_0)$ be the intersection of L_1 and L_3, where t_0 is to be determined. Since $\overrightarrow{QR} \perp L_1$,

$$0 = \overrightarrow{QR} \cdot \mathbf{v} = \langle 2t_0, -1 - t_0, 5 + t_0 \rangle \cdot \langle 2, -1, 1 \rangle = 6t_0 + 6,$$

hence $t_0 = -1$, and $R = (-1, 3, 2)$, $\overrightarrow{RQ} = \langle -2, 0, 4 \rangle$.

$$\text{Vector Equation for Line } L_3: \ \mathbf{r} = \langle 1, 3, -2 \rangle + t\langle -2, 0, 4 \rangle.$$

Relation Between Two Lines

In $\mathbb{R}^2$, two lines are either parallel or intersecting; While in $\mathbb{R}^3$, two lines can be parallel, intersecting, or skew (neither parallel nor intersecting).

Example

Determine the relation between

$$L_1: \quad x = 1 + t,\ y = -2 + 3t,\ z = 4 - t.$$

$$L_2: \quad x = 2s,\ y = 3 + s,\ z = -3 + 4s.$$

Solution: L_1 and L_2 are NOT parallel since their direction vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are NOT parallel. Combining two parametric equations,

$$1 + t = 2s,\ -2 + 3t = 3 + s,\ 4 - t = 3 + 4s$$

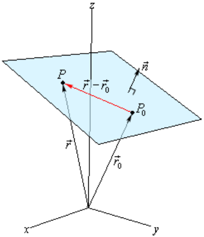
gives no solution, which indicates that these two lines do NOT intersect. Therefore, L_1 and L_2 are skew lines.

Vector Equation of Planes

Consider a plane through a point P_0 (reference point) and perpendicular to a vector $\mathbf{n}$, called the **normal vector**. Given an arbitrary point P on the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. The **vector equation** of the plane is given by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Proof. Since $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$ is orthogonal to $\mathbf{n}$.



Scalar and Linear Equation of Planes

If the points $P = (x, y, z)$, $P_0 = (x_0, y_0, z_0)$ and the normal vector $\mathbf{n} = \langle a, b, c \rangle$, we rewrite the vector equation

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0,$$

$$\textit{or } \textbf{Scalar Equation}: \; a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Taking $d = -ax_0 - by_0 - cz_0$, we obtain

$$\textbf{Linear Equation}: \; ax + by + cz + d = 0.$$

Example

Find the scalar and linear equation of the plane through the points $P(1,3,2)$, $Q(3,-1,6)$ *and* $R(5,2,0)$.

Solution. The vectors

$$\overrightarrow{PQ} = \langle 2, -4, 4 \rangle, \quad \overrightarrow{PR} = \langle 4, -1, -2 \rangle$$

both lie in the plane, and hence the normal vector

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}.$$

Take $P(1,3,2)$ as a reference point.

Scalar Equation : $12(x-1) + 20(y-3) + 14(z-2) = 0.$

Linear Equation : $12x+20y+14z = 100$ or $6x+10y+7z = 50.$

Relation Between A Line and A Plane

Definition

Let $\mathbf{v}$ *be the direction vector of a line, and* $\mathbf{n}$ *the normal vector to a plane. The line and the plane are*

$$\begin{cases} \textit{parallel}, & \textit{if } \mathbf{v} \perp \mathbf{n} \\ \textit{intersecting at a point}, & \textit{otherwise}. \end{cases}$$

Algebraically, we can combine the equations of the line and the plane and try to solve them. The solutions give us the intersection point, while no solution means parallel.

Example

Find the point at which the line

$$x = 2 + 3t, \; y = -4t, \; z = 5 + t$$

intersects the plane $4x + 5y - 2z = 18$.

Solution.

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18,$$

so $t = -2$ and $x = -4, y = 8, z = 3$, thus the point of intersection is $(-4, 8, 3)$.

Example

Show that the line

$$x = 2 + 3t,\ \ y = -4t,\ \ z = 5 - 4t$$

is parallel to the plane $4x + 5y - 2z = 18$.

Solution 1.

$$4(2 + 3t) + 5(-4t) - 2(5 - 4t) = 18,$$

after simplification, we get $-2 = 18$. So the line is parallel to the plane.

Solution 2. Direction Vector of the line is $\mathbf{v} = \langle 3, -4, -4 \rangle$, and the normal vector to the plane is $\mathbf{n} = \langle 4, 5, -2 \rangle$. $\mathbf{v} \perp \mathbf{n}$ since

$$\mathbf{v} \cdot \mathbf{n} = 3 \cdot 4 + (-4) \cdot 5 + (-4) \cdot (-2) = 0.$$

Relation Between Two Planes

Definition

Let $\mathbf{n}_1, \mathbf{n}_2$ *be the normal vectors to two planes. The planes are*

$$\begin{cases} \textit{parallel}, & \textit{if } \mathbf{n}_1 \parallel \mathbf{n}_2 \\ \textit{intersecting at a line}, & \textit{otherwise}. \end{cases}$$

The angle between these two planes is the angle between $\mathbf{n}_1, \mathbf{n}_2$*.*

Algebraically, we can combine the equations of the planes and try to solve them. The solutions give us the line of intersection, while no solution means parallel.

Example

Given two planes $2x + y - 3z = 2$ *and* $-x + 2y - z = 1$. *Find the angle in between and the line of intersection.*

Solution. $\mathbf{n}_1 = \langle 2, 1, -3 \rangle$ and $\mathbf{n}_2 = \langle -1, 2, -1 \rangle$.

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{3}{\sqrt{84}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{3}{\sqrt{84}}\right)$$

For the intersection line:

Geometric method: find a point P satisfying both equations, say $P = (3/5, 4/5, 0)$. The direction vector of intersection line is

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ -1 & 2 & -1 \end{vmatrix} = -5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k},$$

or simply, take $\mathbf{v} = \langle 1, 1, 1 \rangle$. So the line is

$$\begin{cases} x = 3/5 + t, \\ y = 4/5 + t, \\ z = \quad\ \ t. \end{cases}$$

Algebraic method: Set $z = t$ and solve x, y in

$$\begin{cases} 2x + y - 3t = 2, \\ -x + 2y - t = 1, \end{cases}$$

in terms of t, get $x = 3/5 + t$, $y = 4/5 + t$.

Distance From A Point to A Plane

Theorem

The distance between the point $P_1 = (x_1, y_1, z_1)$ *and the plane* $ax + by + cz + d = 0$ *is given by*

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Proof. Choose a point $P_0 = (x_0, y_0, z_0)$ on the plane. The distance is given by the absolute value of the scalar projection of $\overrightarrow{P_0P}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$, and hence

$$D = |comp_{\mathbf{n}} \overrightarrow{P_0P}| = \frac{|\mathbf{n} \cdot \overrightarrow{P_0P}|}{|\mathbf{n}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

Using $ax_0 + by_0 + cz_0 = d$, we get the desired formula.

Corollary

The distance between two parallel planes

$$ax + by + cz + d_1 = 0, \quad and \quad ax + by + cz + d_2 = 0.$$

is given by

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof. Take a point $P_1(x_1, y_1, z_1)$ on the second plane. Since $ax_1 + by_1 + cz_1 + d_2 = 0$, the distance between these two planes is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example

Find the distance between the parallel planes

$$10x + 2y - 2z = 5, \quad \textit{and} \;\; 5x + y - z = 1$$

Solution. Rewrite the above equations as

$$10x + 2y - 2z - 5 = 0, \quad textand \;\; 10x + 2y - 2z - 2 = 0,$$

then the distance is given by

$$\frac{|-5-(-2)|}{\sqrt{10^2 + 2^2 + (-2)^2}} = \frac{3}{\sqrt{108}} = \frac{\sqrt{3}}{6}.$$

Example

Find the distance between the skew lines

$$L_1: \quad x = 1 + t, \; y = -2 + 3t, \; z = 4 - t.$$

$$L_2: \quad x = 2s, \; y = 3 + s, \; z = -3 + 4s.$$

Solution. Take a point from L_1, say $P_1 = (1, -2, 4)$, and a point from L_2, say $P_2 = (0, 3, -3)$. The direction vector for L_1 is $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$, and for L_2 is $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$, then the vector

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

Then the distance between these two lines is given by the absolute value of scalar projection of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$, that is,

$$D = |comp_{\mathbf{n}} \overrightarrow{P_1P_2}| = \frac{|\mathbf{n} \cdot \overrightarrow{P_1P_2}|}{|\mathbf{n}|} = \frac{|\langle 13, -6, -5 \rangle \cdot \langle -1, 5, -7 \rangle|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}}.$$