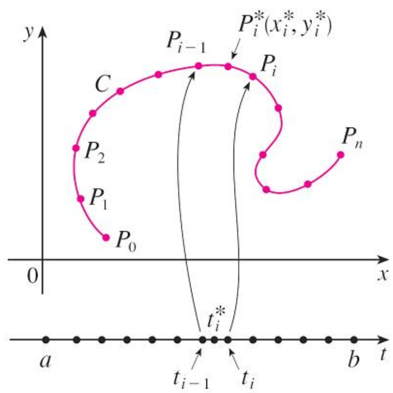
1. **Line Integral with respect to the arc length:** Consider a planar curve C given by

$$\mathbf{r}(t) = \langle x(t),\ y(t) \rangle, \quad a \le t \le b.$$

Divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$, $i = 0, \dots, n$, and thus divide the curve C correspondingly n sub-curves of length Δs_i. Choose randomly a point $P_i^* = (x_i^*, y_i^*)$ on the i-th sub-curve.



Given a function $f(x, y)$, consider the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i.$$

Definition 1: the **line integral of f along C (with respect to arc length)** is

$$\int_C f(x,y) ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

provided the limit exists.

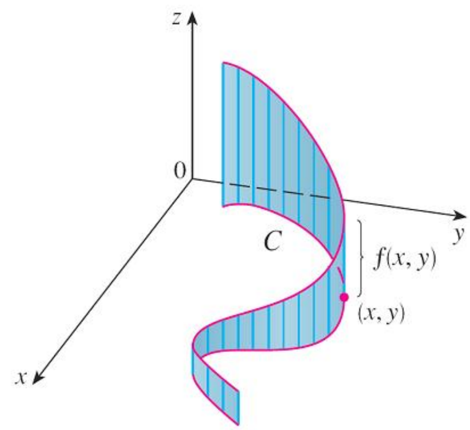
Since

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{x'(t)^2 + y'(t)^2}\, dt,$$

Theorem 1:

$$\int_C f(x,y) ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Geometric Meaning of Line Integral:



If $f(x, y) \geq 0$, then the line integral

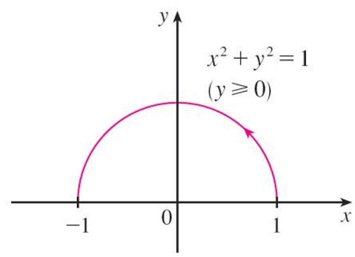
$$\int_C f(x, y) ds$$

represents the area of one side of the "fence" or "curtain" in the above figure, whose base is C and whose height above the point (x, y) is $f(x, y)$.

Example 1: Evaluate

$$\int_C (2 + x^2 y)\ ds,$$

where C is the upper half of the unit circle $x^2 + y^2 = 1$.



Solution: the half circle can be parametrized by

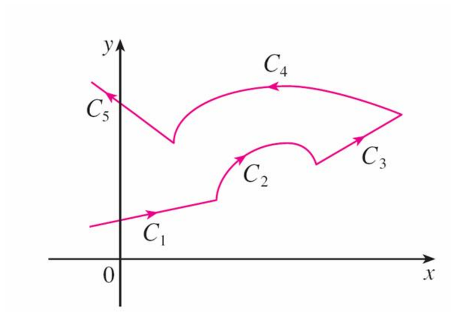
$$\begin{cases} x = \cos t, \\ y = \sin t, \end{cases} \qquad 0 \le t \le \pi.$$

Then

$$\begin{aligned}
\int_C (2 + x^2 y)\ ds &= \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{[(\cos t)']^2 + [(\sin t)']^2} dt \\
&= \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{[-\sin t]^2 + [\cos t]^2} dt \\
&= \int_0^\pi (2 + \cos^2 t \sin t) dt \\
&= \left[2t - \frac{1}{3}\cos^3 t \right]_0^\pi \\
&= (2\pi + \frac{1}{3}) - (-\frac{1}{3}) = 2\pi + \frac{2}{3}.
\end{aligned}$$

In general, the curve C might be **piecewise smooth**, that is, C is a union of a finite number of smooth curves $C_1, \dots, C_n$, then

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \cdots + \int_{C_n} f(x,y)ds.$$



Example 2: Evaluate $\int_C 2xds$, where C consists of the are C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Solution: $\int_C 2xds = \int_{C_1} 2xds + \int_{C_2} 2xds$, where

(1) the first curve C_1 is given by

$$x = x, \ \ y = x^2, \quad 0 \le x \le 1,$$

$$\begin{aligned} \int_{C_1} 2xds &= \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_0^1 2x\sqrt{1+4x^2}dx \\ &= \frac{1}{6}(1+4x^2)^{3/2}\Big|_0^1 = \frac{5\sqrt{5}-1}{6} \end{aligned}$$

(2) the second curve C_2 is given by

$$x = 1, \ \ y = y, \quad 1 \le y \le 2,$$

$$\begin{aligned} \int_{C_2} 2xds &= \int_1^2 2\cdot 1\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dy \\ &= \int_1^2 2\sqrt{0+1^2}dy = \int_1^2 2dy = 2. \end{aligned}$$

Therefore,

$$\int_C 2xds = \int_{C_1} 2xds + \int_{C_2} 2xds = \frac{5\sqrt{5}-1}{6} + 2.$$

2. **Line integral with respect to x and y:** In the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i,$$

we can replace Δs_i by either Δx_i or Δy_i, then

Definition 2: the **line integral of f along C with respect to x and y** is

$$\int_C f(x,y)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i$$

$$\int_C f(x,y)dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i$$

provided the limits exists.

Since $dx = x'(t)dt$ and $dy = y'(t)dt$, then

Theorem 2:

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))x'(t)dt.$$

$$\int_C f(x,y)dy = \int_a^b f(x(t), y(t))y'(t)dt.$$

It frequently happens that line integrals with respect to x and y occur together along the same curve C. When this happens, it's customary to abbreviate writing

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy.$$

Remark: There are two **orientations** for a given curve. If we denote a curve by C with certain orientation, we shall denote by $-C$ the same curve but of reverse orientation.

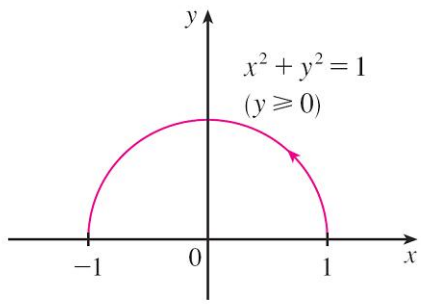
(1) The orientation does not affect line integral w.r.t. the arc length, that is,

$$\int_{-C} f(x,y)ds = \int_C f(x,y)ds;$$

(2) Reverse orientation gives opposite values for line integral w.r.t. x and y, that is,

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx, \quad \int_{-C} f(x,y)dy = -\int_C f(x,y)dy.$$

Example 3: Evaluate
$$\int_C (1+y)dx + xdy,$$
where C is the arc of the upper half unit circle from $(1,0)$ to $(-1,0)$.



Solution: the half circle from $(1,0)$ to $(-1,0)$ can be parametrized by
$$\begin{cases} x = \cos t, \\ y = \sin t, \end{cases} \qquad 0 \le t \le \pi.$$

Then
$$\begin{aligned}
\int_C (1+y)dx + xdy &= \int_0^\pi (1+\sin t)(\cos t)'dt + \cos t(\sin t)'dt \\
&= \int_0^\pi (1+\sin t)(-\sin t)dt + \cos t(\cos t)dt \\
&= \int_0^\pi [-\sin t + (\cos^2 t - \sin^2 t)]dt \\
&= \int_0^\pi [-\sin t + \cos(2t)]dt \\
&= \cos t + \frac{1}{2}\sin(2t)\Big|_0^\pi \\
&= [\cos\pi + \frac{1}{2}\sin(2\pi)] - [\cos 0 + \frac{1}{2}\sin 0] = -2.
\end{aligned}$$

Remark: The reverse curve $-C$ is the arc of the upper half unit circle from $(-1,0)$ to $(1,0)$, that is,
$$\begin{cases} x = \cos t, \\ y = \sin t, \end{cases} \qquad t \text{ varies from } \pi \text{ to } 0,$$
so $\int_{-C}(1+y)dx + xdy$ is the integral from π to 0, which equals to 2 (after calculation).

3. **Line integral in Space:** Consider a space curve C given by

$$\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle, \quad a \le t \le b.$$

(1) **Line integral of** $f(x,y,z)$ **along** C **with respect to arc length** is

$$\begin{aligned}\int_C f(x,y,z)ds &= \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt \\ &= \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt.\end{aligned}$$

In particular, the length of the curve C is given by $L = \int_C 1ds = \int_a^b |\mathbf{r}'(t)|dt$.

(2) **Line integral of** $f(x,y,z)$ **along** C **with respect to** x**,** y **and** z is

$$\begin{aligned}\int_C f(x,y,z)dx &= \int_a^b f(x(t),y(t),z(t))x'(t)dt, \\ \int_C f(x,y,z)dy &= \int_a^b f(x(t),y(t),z(t))y'(t)dt, \\ \int_C f(x,y,z)dz &= \int_a^b f(x(t),y(t),z(t))z'(t)dt,\end{aligned}$$

Or combinatorially,

$$\begin{aligned}&\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz \\ = &\int_a^b [P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t))y'(t) + R(x(t),y(t),z(t))z'(t)]dt\end{aligned}$$

Example 4: Let C be the circular helix given by the equation

$$x = \cos t,\ y = \sin t,\ z = t, \quad 0 \le t \le \frac{\pi}{4}.$$

(1) Evaluate $\int_C y \sin z ds$.

(2) Evaluate $\int_C y \sin z dx + x \sin z dy + xy \cos z dz$.

Solution:

(1)

$$\begin{aligned}
\int_C y \sin z ds &= \int_0^{\frac{\pi}{4}} \sin t \sin t \sqrt{[(\cos t)']^2 + [(\sin t)']^2 + [(t)']^2}\, dt \\
&= \int_0^{\frac{\pi}{4}} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} dt \\
&= \int_0^{\frac{\pi}{4}} \sin^2 t \sqrt{2} dt \\
&= \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{1 - \cos(2t)}{2}\, dt \\
&= \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin(2t) \right]_0^{\frac{\pi}{4}} \\
&= \frac{\sqrt{2}}{2} \left[\left(\frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} \right) - 0 \right] = \frac{\sqrt{2}(\pi - 2)}{8}
\end{aligned}$$

(2)

$$\begin{aligned}
&\int_C y \sin z dx + x \sin z dy + xy \cos z dz \\
= &\int_0^{\frac{\pi}{4}} [\sin t \sin t (\cos t)' + \cos t \sin t (\sin t)' + \cos t \sin t \cos t (t)']\, dt \\
= &\int_0^{\frac{\pi}{4}} (-\sin^3 t + 2 \sin t \cos^2 t) dt \\
= &\int_0^{\frac{\pi}{4}} \sin t [3 \cos^2 t - 1] dt \\
= &\left[\cos t - \cos^3 t \right]_0^{\frac{\pi}{4}} = \left[\cos \frac{\pi}{4} - \cos^3 \left(\frac{\pi}{4} \right) \right] - [\cos 0 - \cos^3 0] = \frac{\sqrt{2}}{4}.
\end{aligned}$$

4. **Line integral of vector fields:** One can regard

$$\int_C P(x,y)dx + Q(x,y)dy \tag{1}$$

as follows. The planar curve C is given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, or simply $\mathbf{r} = \langle x, y\rangle$, and thus $d\mathbf{r} = \langle dx, dy\rangle$. Denote the vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, or simply $\mathbf{F} = \langle P, Q\rangle$. Therefore, we interpret (1) into

Line integral of the vector field F along C**:**

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt = \int_C P(x,y)dx + Q(x,y)dy$$

Similarly, one regard

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz \tag{2}$$

as **Line integral of the vector field F along a space curve** C**:**

$$\int_C \mathbf{F}\cdot dr = \int_C \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt = \int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz,$$

where the curve C is given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $d\mathbf{r} = \langle dx, dy, dz\rangle$, and the vector field $\mathbf{F} = \langle P, Q, R\rangle$.

As an application, the total work done by the force $\mathbf{F} = \langle P, Q, R\rangle$ in moving the particle along C given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ can be computed by

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz.$$

Example 5: Find the work done by the force field

$$\mathbf{F}(x, y, z) = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$$

in moving a particle along the twisted cubic curve C given by

$$x = t,\ y = t^2,\ z = t^3, \quad 0 \le t \le 1.$$

Solution. Along the curve C,

$$\mathbf{F} = t^2t^3\mathbf{i} + t^3zt\mathbf{j} + tt^2\mathbf{k} = t^5\mathbf{i} + t^4\mathbf{j} + t^3\mathbf{k} = \langle t^5, t^4, t^3\rangle,$$

and

$$d\mathbf{r} = \langle dx, dy, dz\rangle = \langle x'(t), y'(t), z'(t)\rangle dt = \langle 1, 2t, 3t^2\rangle dt,$$

then

$$\begin{aligned} W = \int_C \mathbf{F} \cdot dr &= \int_0^1 \langle t^5, t^4, t^3\rangle \cdot \langle 1, 2t, 3t^2\rangle dt \\ &= \int_0^1 6t^5 dt = t^6|_0^1 = 1. \end{aligned}$$