CURVATURE MEASURES(1)

BY
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1. Introduction. In the classical theory of *convex subsets of Euclidean n space* [BF; H] a major role is played by Minkowski's *Quermassintegrale*. These are, up to constant factors, the coefficients of the *Steiner polynomial* whose value at any positive number r equals the n dimensional measure of the r neighborhood of the convex set considered. For a set with sufficiently smooth boundary, they may be computed by integrating the *symmetric functions of the principal curvatures* over the bounding hypersurface.

In the branch of classical differential geometry known as *integral geometry* [BE; S; C2] similar concepts have been studied *without convexity assumption* for certain types of sets, for example regions bounded by very smooth hypersurfaces. The central result of this study is the *principal kinematic formula* for the integral, over the group of rigid motions of n space, of the Euler-Poincaré characteristic of the intersection of two solid bodies, one fixed and the other moving.

In [W] the formula of Steiner was extended to compact regular *submanifolds* of class 2 of n space, with coefficients expressed as integrals over the manifold of certain scalars associated with the *Riemannian curvature tensor*. This work was followed by the generalization of the *Gauss-Bonnet Theorem* [A; FE1; AW; C1].

All these classical investigations involve related geometric and measure theoretic curvature properties of various special types of point sets. The search for a general theory is an obvious challenge. Those subsets of n space which are to be the objects of such a theory must be singled out by some simple geometric property. Among these objects must be all convex sets and all regularly embedded manifolds of class 2 (possibly with regular boundary). The curvatures attached to these objects should have the global aspects of Minkowski's Quermassintegrale, yet be determined by local properties; hence it seems reasonable that they should be measures. Neither the definition of the curvature measures nor the statement of any important theorem about them may contain explicit assumptions of differentiability, because arbitrary convex sets are to be admissible objects. Whatever differentiability may be required for an auxiliary analytic or algebraic argument must be implied by

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the geometric properties. Of course, in order to be worth while, such a theory must contain natural generalizations of the principal kinematic formula and of the Gauss-Bonnet Theorem.

This problem presents a timely challenge to a worker in modern real function theory, which was originally created in large part for the study of geometric questions. The results of the theory of area have greatly contributed to the understanding of first order tangential properties of point sets, and one can hope for similar success in dealing with second order differential geometric concepts such as curvature. In particular the author's previous work connecting Hausdorff measure with various integral geometric formulae [F3, 4, 5, 7] may be considered a first order antecedent of the second order theory developed in this paper.

The objects treated here are the *sets with positive reach*; the reach of a subset A of Euclidean n space, E_n, is the largest ϵ (possibly ∞) such that if $x \in E_n$ and the distance, $\delta_A(x)$, from x to A is smaller than ϵ, then A contains a unique point, $\xi_A(x)$, nearest to x. Assuming that $\text{reach}(A) > 0$, Steiner's formula is established in the following form: For each bounded Borel subset Q of E_n and for $0 \leqq r < \text{reach}(A)$, the n dimensional measure of

$$E_n \cap \{x \colon \delta_A(x) \leqq r \text{ and } \xi_A(x) \in Q\}$$

is given by a polynomial of degree at most n in r, say

$$\sum_{i=0}^{n} r^{n-i} \alpha(n-i) \Phi_i(A, Q)$$

where $\alpha(j)$ is the j dimensional measure of a spherical ball with radius 1 in E_j. Clearly the coefficients $\Phi_i(A, Q)$ are countably additive with respect to Q, defining the *curvature measures*

$$\Phi_0(A, \cdot), \Phi_1(A, \cdot), \cdots, \Phi_n(A, \cdot).$$

If $\dim A = k$, then $\Phi_i(A, \cdot) = 0$ for $i > k$, $\Phi_k(A, \cdot)$ is the restriction of the k dimensional Hausdorff measure to A, and the measures $\Phi_i(A, \cdot)$ corresponding to $i < k$ depend on second order properties of A. If a sequence of sets, all with reach at least $\epsilon > 0$, is convergent relative to the Hausdorff metric, then the associated sequences of curvature measures converge weakly to the curvature measures of the limit set, whose reach is also at least ϵ. In this way any set A with positive reach may be approximated in curvature by the solids

$$\{x \colon \delta_A(x) \leqq s\}$$

corresponding to $s > 0$. If A, B and $A \cup B$ have positive reach, so does $A \cap B$, and

$$\Phi_i(A, \cdot) + \Phi_i(B, \cdot) = \Phi_i(A \cup B, \cdot) + \Phi_i(A \cap B, \cdot).$$

If $A \subset E_m$ and $B \subset E_n$ have positive reach, so does $A \times B \subset E_m \times E_n \equiv E_{m+n}$, and

$$\Phi_k(A \times B, \cdot) = \sum_{i+j=k} \Phi_i(A, \cdot) \otimes \Phi_j(B, \cdot)$$

where $\otimes$ is the cartesian product of measures. The Gauss-Bonnet Theorem generalizes to the proposition that if A is a compact set with positive reach, then the total curvature $\Phi_0(A, A)$ equals the Euler-Poincaré characteristic of A. The new version of the principal integralgeometric formula states that if μ is a Haar measure of the group of isometries of E_n, A and B are subsets of E_n with positive reach, and B is compact, then $A \cap g(B)$ has positive reach for μ almost all isometries g, and

$$\int \Phi_i[A \cap g(B), \chi \cdot (\psi \circ g^{-1})] d\mu g$$
$$= \sum_{k+l=n+i} c_{n,k,l} \Phi_k(A, \chi) \Phi_l(B, \psi)$$

whenever χ and ψ are bounded Baire functions on E_n, χ with bounded support; here $c_{n,k,l}$ are constants determined by the choice of μ.

Analytic methods can be used in the proof of some of these geometric theorems, because the concept of reach of a set A is closely related to differentiability properties of the functions δ_A and ξ_A. In fact, reach$(A) \geqq \epsilon$ if and only if δ_A is continuously differentiable on $\{x: 0 < \delta_A(x) < \epsilon\}$. Furthermore, if reach$(A) > s > t > 0$, then grad δ_A is Lipschitzian on $\{x: t \leqq \delta_A(x) \leqq s\}$, and ξ_A is Lipschitzian on $\{x: \delta_A(x) \leqq s\}$; hence $\{x: \delta_A(x) = s\}$ is an $n-1$ dimensional manifold of class 1, with Lipschitzian normal, whose second fundamental form exists almost everywhere.

The computations involving curvature tensors are greatly simplified through use of the algebra $\wedge^*(E) \otimes \wedge^*(E)$ and its trace function; here $\wedge^*(E)$ is the covariant exterior algebra of a vectorspace E. A similar algebra has been used in [FL1, 2].

The paper contains a new integral formula concerning Hausdorff measure, which is used here in the proof of the principal kinematic formula, but which also has other applications. Suppose X and Y are m and k dimensional Riemannian manifolds of class 1, $m \geqq k$, and $f: X \to Y$ is a Lipschitzian map. For $y \in Y$ compute the $m-k$ dimensional Hausdorff measure of $f^{-1}\{y\}$, and integrate over Y with respect to k dimensional Hausdorff measure. It is shown that this integral equals the integral over X, with respect to m dimensional Hausdorff measure, of the Jacobian whose value at x is the norm of the linear transformation of k-vectors induced by the differential of f at x. This result is the counterpart of the classical integral formula for area, which deals with the case when $m \leqq k$.

2. **Some definitions.** The purpose of 2.1 to 2.9 is only to fix notations concerning certain well known concepts; more details may be found in references such as [S] and [B2] regarding 2.3, [L2] regarding 2.4, [F4] regarding 2.6 and 2.7, [L1] regarding 2.8, and [B1], [W2] or [F9] regarding 2.9. Some new material occurs in 2.10 to 2.13.

2.1. Definition. E_n is the *n dimensional Euclidean space* consisting of all sequences $x=(x_1, \cdots, x_n)$ of real numbers, with the *inner product*

$$x \bullet y = \sum_{i=1}^{n} x_i y_i \qquad \text{for } x, y \in E_n.$$

G_n is the *orthogonal group* of E_n. With $z \in E_n$ associate the *translation*

$$T_z\colon E_n \to E_n,\ T_z(x) = z + x \qquad \text{for } x \in E_n.$$

With $R \in G_n$ and $w \in E_{n-m}$ associate the *m dimensional plane*

$$\lambda_n^m(R, w) = R(E_n \cap \{x : x_i = w_i \text{ for } i = 1, \cdots, n-m\}).$$

2.2. Definition. Suppose f maps an open subset of E_n into E_m.

If f is *differentiable* (in the sense of Fréchet) at x, then the differential $Df(x)$ is the linear transformation of E_n into E_m characterized by the equation

$$\lim_{h \to 0} \left| f(x+h) - f(x) - [Df(x)](h) \right| / \left| h \right| = 0.$$

In case $m=1$, $\operatorname{grad} f(x) \in E_n$ is characterized by the property that

$$[Df(x)](h) = [\operatorname{grad} f(x)] \bullet h \qquad \text{for } h \in E_n.$$

For $i=1, \cdots, n$, $D_i f(x)$ is the partial derivative of f at x in the direction of the vector whose coordinates are 0 except the ith, which equals 1.

2.3. Definition. Use will be made both of *Carathéodory outer measures* [S, Chapter 2] and of countably additive functions [S, Chapter 1] on the class of all Borel sets with compact closure in a locally compact space, which will be called *Radon measures* in accordance with [B2, Chapter 3]. A measure μ over a space X may be thought of either as a function on a suitable class of subsets of X, or as a function on a suitable class of functions on X. It is convenient to use the alternate notations

$$\int_X f(x)\, d\mu x = \int f d\mu = \mu(f).$$

With each Radon measure μ one associates its *variation measure* $|\mu|$.

With measures μ and ν over X and Y one associates the *cartesian product measure* $\mu \otimes \nu$ over $X \times Y$.

2.4. Definition. L_n is the *n dimensional Lebesgue measure* over E_n. ϕ_n is the *Haar measure of* G_n such that $\phi_n(G_n)=1$.

Under the map which associates with $(z, R) \in E_n \times G_n$ the isometry $T_z \circ R$ of E_n, the image of the measure $L_n \otimes \phi_n$ is a *Haar measure of the group of isometries of* E_n.

Under the map λ_n^m, the image of the measure $\phi_n \otimes L_{n-m}$ is a *Haar measure for the space of all m dimensional planes in* E_n, invariant under the group of isometries of E_n.

2.5. DEFINITION.

$$\alpha(k) = L_k(E_k \cap \{x : |x| \leqq 1\}) = 2^k \Gamma\left(\frac{1}{2}\right)^{k-1} \Gamma\left(\frac{k+1}{2}\right) \Gamma(k+1)^{-1},$$

$$\beta(n,k) = \frac{\alpha(k)\,\alpha(n-k)}{\alpha(n)\binom{n}{k}} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)},$$

$$\gamma(n,k,l) = \frac{\beta(n,k)\beta(n,l)}{\beta(n,k+l-n)\beta(2n-k-l,n-l)} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{k+l-n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}.$$

2.6. DEFINITION. H^k is the *k dimensional Hausdorff measure.* If A is a subset of a metric space, then $H^k(A)$ equals the limit, as $r \to 0+$, of the infimum of the sums

$$\sum_{S \in F} 2^{-k}\alpha(k) \text{ diameter}(S)^k$$

corresponding to all countable coverings F of A such that diameter$(S) < r$ for $S \in F$.

2.7. DEFINITION. A subset of a metric space is called *k rectifiable* if and only if it is the image of a bounded subset of E_k under a Lipschitzian map. The union of a countable family of k rectifiable sets is said to be *countably k rectifiable.*

2.8. REMARK. Since differentiability is invariant under continuously differentiable homeomorphisms, this concept remains meaningful for maps of manifolds of class 1. An intrinsic tangent vector v of a manifold of class 1 at a point p may be thought of as operating on every function f which maps some neighborhood of p into some Euclidean space and which is differentiable at p; then $df(v) = v(f)$.

In generalizing measure theoretic properties from E_n to an n-dimensional Riemannian manifold of class 1, one replaces L_n by H^n. A Lipschitzian map of such a manifold into some other Riemann manifold is differentiable H^n almost everywhere.

If X and Y are m and n dimensional Riemannian manifolds of class 1, then

$$H^{m+n}(S) = (H^m \otimes H^n)(S) \qquad \text{for } S \subset X \times Y.$$

Using matrices, one may think of G_n as an $n(n-1)/2$ dimensional compact submanifold of E_{n^2}. Then the left and right translations of G_n are induced by elements of G_{n^2}, hence $H^{n(n-1)/2}$ induces a Haar measure over G_n, and

$$H^{n(n-1)/2}(S) = H^{n(n-1)/2}(G_n)\cdot\phi_n(S) \qquad \text{for } S \subset G_n.$$

2.9. DEFINITION. For each finite dimensional real vectorspace E and $k=0, \cdots, \dim E$ let

$$\bigwedge_k(E) \quad \text{and} \quad \bigwedge^k(E)$$

be the associated spaces of *k-vectors* and *k-covectors* (contravariant and covariant skewsymmetric tensors of rank k). Also let

$$\bigwedge_*(E) = \bigoplus_{k=0}^{\dim E} \bigwedge_k(E) \quad \text{and} \quad \bigwedge^*(E) = \bigoplus_{k=0}^{\dim E} \bigwedge^k(E)$$

be the corresponding *exterior algebras*, with the Grassman multiplication $\wedge$.

With each inner product of E one associates the unique inner products of $\bigwedge_*(E)$ and $\bigwedge^*(E)$ such that the Grassman products of the subsets of any orthonormal base of E form an orthonormal base of $\bigwedge_*(E)$, and the Grassman products of the subsets of the dual base of $\bigwedge^1(E)$ form an orthonormal base of $\bigwedge^*(E)$.

2.10. DEFINITION. Suppose X and Y are Riemannian manifolds of class 1, $f: X\rightarrow Y$, and

$$k = \inf\{\dim X, \dim Y\}.$$

If $p\in X$, f is differentiable at p, E and F are the tangent spaces of X and Y at p and $f(p)$, then the differential of f induces dual linear transformations of $\bigwedge_k(E)$ into $\bigwedge_k(F)$ and of $\bigwedge^k(F)$ into $\bigwedge^k(E)$ with the common norm

$$Jf(p).$$

Using the matrix of the differential of f at p with respect to orthonormal bases for E and F, one computes $Jf(p)$ as the square root of the sum of the squares of the determinants of the k by k minors of this matrix.

2.11. DEFINITION. Suppose E is an n dimensional real vector space. Consider the tensor products

$$\bigwedge^{k,l}(E) = \bigwedge^k(E) \otimes \bigwedge^l(E) \qquad \text{for } k, l = 0, 1, \cdots, n,$$

$$\bigwedge^{**}(E) = \bigwedge^*(E) \otimes \bigwedge^*(E) = \bigoplus_{k,l=0}^{n} \bigwedge^{k,l}(E)$$

and make $\bigwedge^{**}(E)$ into an associative algebra by defining the product

$$(a \otimes b)\cdot(c \otimes d) = (a \wedge c) \otimes (b \wedge d) \qquad \text{for } a, b, c, d \in \bigwedge^*(E).$$

Observe that while $\bigwedge^*(E)$ is anticommutative, $\bigwedge^{**}(E)$ is not anticommutative. However this definition has the advantage that the subalgebra

$$\bigoplus_{k=0}^{n} \wedge^{k,k}(E)$$

is commutative.

This construction is natural; a linear transformation $f: E \to F$ induces a homomorphism $f^*: \wedge^{**}(F) \to \wedge^{**}(E)$.

Now fix an inner product $\bullet$ of E. The corresponding inner product $\bullet$ of $\wedge^*(E)$ induces a unique inner product $\bullet$ of $\wedge^{**}(E)$ such that

$$(a \otimes b) \bullet (c \otimes d) = (a \bullet c)(b \bullet d) \qquad \text{for } a, b, c, d \in \wedge^*(E).$$

On the other hand the inner product of $\wedge^*(E)$ corresponds to a real valued linear function on $\wedge^{**}(E)$, the *trace*, which is characterized by the formula

$$\operatorname{trace}(a \otimes b) = a \bullet b \qquad \text{for } a, b \in \wedge^*(E).$$

Since $\wedge^k(E)$ is the conjugate space of $\wedge_k(E)$, there is a natural isomorphism of $\wedge^{k,k}(E)$ onto the space of bilinear forms of $\wedge_k(E)$; if $a, b \in \wedge^k(E)$, then the bilinear form B corresponding to $(a \otimes b)$ is given by the equation

$$B(x, y) = a(x)b(y) \qquad \text{for } x, y \in \wedge_k(E).$$

Furthermore the space of bilinear forms of $\wedge_k(E)$ is isomorphic with the space of endomorphisms of $\wedge_k(E)$; a bilinear form B and the corresponding endomorphism T are related by the formula

$$B(x, y) = T(x) \bullet y \qquad \text{for } x, y \in \wedge_k(E).$$

In particular, if $\theta_1, \cdots, \theta_n$ form an orthonormal base of $\wedge^1(E)$ and

$$I = \sum_{i=1}^{n} \theta_i \otimes \theta_i \in \wedge^{1,1}(E),$$

then the inner product and the identity endomorphism correspond to I.

2.12. Remark. Assume the conditions of 2.11 and suppose $\theta_1, \cdots, \theta_n$ form an orthonormal base of $\wedge^1(E)$. For $k = 0, \cdots, n$ let S_k be the class of all subsets of $\{1, \cdots, n\}$ with k elements, and for $a \in S_k$ let

$$\theta_a = \theta_{a_1} \wedge \theta_{a_2} \wedge \cdots \wedge \theta_{a_k}$$

where $a_1 < a_2 < \cdots < a_k$ are the elements of a. Then the following statements hold:

(1) $\{\theta_a \otimes \theta_b : a \in S_i, b \in S_j\}$ is an orthonormal base of $\wedge^{i,j}(E)$.

(2) If $M \in \wedge^{i,j}(E)$ and $N \in \wedge^{k,l}(E)$, then

$$|MN| \leqq \left[\binom{i+k}{i}\binom{j+l}{j}\right]^{1/2} |M| \cdot |N|.$$

(3) If $M \in \wedge^{1,1}(E)$, then $|M^k| \leqq k!|M|^k$.

(4) If $M \in \wedge^{k,k}(E)$ and $j = 0, \cdots, n-k$, then

$$\operatorname{trace}(MI^j) = (n-k)!(n-k-j)!^{-1}\operatorname{trace}(M).$$

(5) If $M \in \Lambda^{1,1}(E)$ and f is the endomorphism of E corresponding to M, then

$$M = \sum_{i=1}^{n} f^*(\theta_i) \otimes \theta_i$$

and the endomorphism of $\Lambda_k(E)$ induced by f corresponds to

$$k!^{-1}M^k = \sum_{a\in S_k} f^*(\theta_a) \otimes \theta_a.$$

Consequently $\det(f) = \operatorname{trace}(n!^{-1}M^n)$ and the characteristic polynomial of f is

$$\operatorname{trace}[n!^{-1}(M-\lambda I)^n] = \sum_{k=0}^{n} \operatorname{trace}(k!^{-1}M^k)\cdot(-\lambda)^{n-k}.$$

(6) If $w_1, \cdots, w_n \in \Lambda^1(E)$ and $M = \sum_{i=1}^n w_i \otimes \theta_i$, then

$$\bigwedge_{i=1}^{n} w_i = \operatorname{trace}(n!^{-1}M^n) \bigwedge_{i=1}^{n} \theta_i.$$

The verification of (1), (4), (5) is quite easy. Furthermore (3) follows by induction from (2), and (6) follows from (5) with $w_i = f^*(\theta_i)$. To prove (2), use (1) to expand

$$M = \sum_{(a,b)\in S_i\times S_j} M_{a,b}\theta_a \otimes \theta_b, \qquad N = \sum_{(c,d)\in S_k\times S_l} N_{c,d}\theta_c \otimes \theta_d.$$

For $(u, v) \in S_{i+k} \times S_{j+l}$ let

$$P(u, v) = (S_i \times S_j \times S_k \times S_1) \cap \{(a, b, c, d): a \cup c = u, b \cup d = v\},$$

and for $(a, b, c, d) \in P(u, v)$ choose $\epsilon_{a,c} = \pm 1$ and $\epsilon_{b,d} = \pm 1$ so that

$$\theta_a \wedge \theta_c = \epsilon_{a,c}\theta_u \text{ and } \theta_b \wedge \theta_d = \epsilon_{b,d}\theta_v.$$

Using (1), Hölder's inequality and the fact that the set $P(u, v)$ are disjoint, one obtains

$$\begin{aligned}
|MN|^2 &= \sum_{(u,v)\in S_{i+k}\times S_{j+l}} \left[\sum_{(a,b,c,d)\in P(u,v)} M_{a,b}N_{c,d}\epsilon_{a,c}\epsilon_{b,d}\right]^2 \\
&\leqq \sum_{(u,v)\in S_{i+k}\times S_{j+l}} \; \sum_{(a,b,c,d)\in P(u,v)} (M_{a,b}N_{c,d})^2 \binom{i+k}{i}\binom{j+l}{j} \\
&\leqq \binom{i+k}{i}\binom{j+l}{j} \sum_{(a,b,c,d)\in S_i\times S_j\times S_k\times S_l} (M_{a,b})^2(N_{c,d})^2 \\
&= \binom{i+k}{i}\binom{j+l}{j} |M|^2|N|^2.
\end{aligned}$$

2.13. REMARK. Under the conditions of 2.11 it is true that *if a real valued linear function Q on $\bigwedge^{k,k}(E)$ is invariant under the endomorphisms of $\bigwedge^{k,k}(E)$ induced by the orthogonal transformations of E, then Q is a real multiple of the trace.*

In fact, using the notations of 2.12, one sees that if $a, b \in S_k$, then

$$Q(\theta_a \otimes \theta_b) = 0 \text{ in case } a \neq b,$$

because if $i \in a - b$ and f is the orthogonal transformation of E such that $f^*(\theta_i) = -\theta_i$ and $f^*(\theta_j) = \theta_j$ for $j \neq i$, then

$$Q(\theta_a \otimes \theta_b) = Q[f^*(\theta_a \otimes \theta_b)] = Q(-\theta_a \otimes \theta_b) = -Q(\theta_a \otimes \theta_b);$$

furthermore

$$Q(\theta_a \otimes \theta_a) = Q(\theta_b \otimes \theta_b)$$

because if $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$ are the elements of a and b, then there is an orthogonal transformation f of E such that

$$f^*(\theta_{a_i}) = \theta_{b_i} \quad \text{for } i = 1, \cdots, k, \quad \text{hence} \quad f^*(\theta_a) = \theta_b.$$

Consequently $Q = Q(\theta_a \otimes \theta_a) \cdot \text{trace}$, where $a \in S_k$.

3. **An integral formula concerning Hausdorff measure**(2). Complementing the classical integral formula for the area of a map $f: X \to Y$ such that $\dim X \leqq \dim Y$, the theorem proved in this section concerns the case when $\dim X \geqq \dim Y$. The original motivation leading to the discovery of this theorem was the simplification of certain arguments in [DG]; in fact, if $X = A = E_m$ and $Y = E_1$, then the formula becomes

$$\int_{E_m} | \operatorname{grad} f(x) | \, dL_m x = \int_{-\infty}^{\infty} H^{m-1}(f^{-1}\{y\}) dy.$$

The theorem will be used in the present paper to prove the kinematic formula, and may be expected to have further applications.

3.1. THEOREM. *If X and Y are separable Riemannian manifolds of class* 1 *with*

$$\dim X = m \geqq k = \dim Y$$

and $f: X \to Y$ is a Lipschitzian map, then

$$\int_A Jf(x) dH^m x = \int_Y H^{m-k}(A \cap f^{-1}\{y\}) dH^k y$$

(2) The author's abstract containing this formula was received by the American Mathematical Society on November 22, 1957, and published in the Notices of the American Mathematical Society as Abstract 542-43 in vol. 5 (1958) p. 167. At the 1958 Summer Institute L. C. Young announced his independent discovery of a very similar theorem and distributed copies of his Technical Summary Report No. 28, U. S. Army Mathematics Research Center, University of Wisconsin, May, 1958, which contains an outline of his argument.

whenever A is an H^m measurable subset of X, and consequently

$$\int_X g(x)Jf(x)dH^m x = \int_Y \int_{f^{-1}\{y\}} g(x)dH^{m-k}xdH^k y$$

whenever g is an H^m integrable function on X.

Proof. Suppose M is a Lipschitz constant for f and let μ be the measure over X such that

$$\mu(A) = \int_Y^* H^{m-k}(A \cap f^{-1}\{y\})dH^k y$$

for $A \subset X$, where "$\int^*$" means upper integral. For $a \in X$, let

$$K(a, r) = X \cap \{x: \text{distance } (x, a) < r\}$$

whenever $r > 0$, and let

$$\mu'(a) = \lim_{r \to 0+} \mu[K(a, r)]/H^m[K(a, r)].$$

The remainder of the argument is divided into seven parts, leading to the first conclusion stated in the theorem. The second conclusion may be derived from the first by the usual algebraic and limit procedure, starting with the case in which g is the characteristic function of an H^m measurable set.

PART 1. *If $A \subset X$, then*

$$\mu(A) \leqq M^k \frac{\alpha(k)\alpha(m-k)}{\alpha(m)} H^m(A).$$

This inequality was proved in [F7, §3].

PART 2. *If A is an H^m measurable subset of X and*

$$v(y) = H^{m-k}(A \cap f^{-1}\{y\})$$

for $y \in Y$, then v is an H^k measurable function.

Proof. If $H^m(A) = 0$ it follows from Part 1 that $v(y) = 0$ for H^k almost all y in Y. Since every H^m measurable subset A of X is the union of an increasing sequence of compact sets and a set of H^m measure zero, it will be sufficient to consider the special case in which A is compact.

For $n = 1, 2, 3, \cdots$ and $y \in Y$ let $v_n(y)$ be the infimum of

$$\sum_{S \in G} 2^{k-m}\alpha(m-k)[\text{diam } (S)]^{m-k}$$

where G is a countable open covering of $A \cap f^{-1}\{y\}$ such that $\text{diam}(S) < n^{-1}$ whenever $S \in G$; then

$$v(y) = \lim_{n \to \infty} v_n(y).$$

Since A is compact, every open covering of $A\cap f^{-1}\{y\}$ is also a covering of $A\cap f^{-1}\{z\}$ provided z is sufficiently close to y. Accordingly the functions v_n are uppersemicontinuous.

PAFT 3. *If A is an H^m measurable subset of X, then*

$$\mu(A) = \int_Y H^{m-k}(A \cap f^{-1}\{y\})dH^k y = \int_A \mu'(x)dH^m x.$$

Proof. The first equation follows from Part 2 and the definition of μ. It shows that μ is completely additive on the class of all H^m measurable subsets of X. On the other hand it follows from Part 1 that μ is absolutely continuous with respect to H^m. Accordingly μ is the indefinite integral of its derivative μ'.

PART 4. *If $a \in X = E_m$, $Y = E_k$, f is continuously differentiable in a neighborhood of a and $Jf(a) = 0$, then $\mu'(a) = 0$.*

Proof. Suppose $\epsilon > 0$. Since range $Df(a) \neq E_k$ there is a real valued linear function q on E_k such that $|q| = 1$ and $q \circ Df(a) = 0$. The continuity of Df at a implies the existence of a convex neighborhood U of a such that

$$| q \circ Df(x) | \leqq \epsilon M \qquad \text{for } x \in U,$$

whence

$$| (q \circ f)(x) - (q \circ f)(z) | \leqq \epsilon M | x - z | \qquad \text{for } x, z \in U.$$

Furthermore, since μ is invariant under rotations of E_k, one may assume that

$$q(y) = y_k \qquad \text{for } y \in E_k.$$

It follows that if S is the endomorphism of E_k such that

$$S(y) = (y_1, \cdots, y_{k-1}, \epsilon^{-1} y_k) \qquad \text{for } y \in E_k,$$

then

$$| (S \circ f)(x) - (S \circ f)(z) | \leqq 2M | x - z | \qquad \text{for } x, z \in U.$$

Applying Part 3 to f and $S \circ f$, and Part 1 to $S \circ f$, one concludes that if A is an H^m measurable subset of U then

$$\begin{aligned} \mu(A) &= \int_{E_k} H^{m-k}(A \cap f^{-1}\{y\})dH^k y \\ &= \int_{E_k} H^{m-k}[A \cap (S \circ f)^{-1}\{S(y)\}]dH^k y \\ &= \epsilon \int_{E_k} H^{m-k}[A \cap (S \circ f)^{-1}\{w\}]dH^k w \\ &\leqq \epsilon (2M)^k \frac{\alpha(k)\alpha(m-k)}{\alpha(m)} H^m(A). \end{aligned}$$

PART 5. *If* $a \in X = E_m$, $Y = E_k$ *and* f *is continuously differentiable in a neighborhood of* a, *then* $\mu'(a) = Jf(a)$.

Proof. In view of Part 4 suppose $Jf(a) \neq 0$. Since

$$\dim \text{kernel } Df(a) = m - k$$

and both Jf and μ' are invariant under rotations of E_m, one may assume that

$$D_i f(a) = 0 \qquad \text{for } i = k + 1, \cdots, m.$$

Letting $F: E_m \to E_m$ be the map such that if $x \in E_m$ then

$$[F(x)]_i = [f(x)]_i \qquad \text{for } i = 1, \cdots, k,$$
$$[F(x)]_i = x_i \qquad \text{for } i = k + 1, \cdots, m,$$

one sees that

$$JF(a) = Jf(a) \neq 0.$$

Accordingly, if r is a small positive number and $A = K(a, r)$, then F is continuously differentiable and univalent on A. For $y \in E_k$ let

$$B_y = E_{m-k} \cap \{z: (y_1, \cdots, y_k, z_1, \cdots, z_{m-k}) \in F(A)\},$$
$$g_y: B_y \to E_m,$$
$$g_y(z) = (F \mid A)^{-1}(y_1, \cdots, y_k, z_1, \cdots, z_{m-k}) \qquad \text{for } z \in B_y,$$

and observe that B_y is open, g_y is continuously differentiable and univalent, with

$$\text{range } g_y = A \cap f^{-1}\{y\}.$$

It follows from the classical formula for area (see [F4, 5.9]) that

$$\begin{aligned}
\mu(A) &= \int_{E_k} H^{m-k}(A \cap f^{-1}\{y\}) dH^k y \\
&= \int_{E_k} \int_{B_y} J g_y(z) dL_{m-k} z dL_k y \\
&= \int_{F(A)} J g_{(w_1, \cdots, w_k)}(w_{k+1}, \cdots, w_m) dL_m w \\
&= \int_A J g_{f(x)}(x_{k+1}, \cdots, x_m) JF(x) dL_m x.
\end{aligned}$$

Accordingly, if r is small, then $\mu(A)/H^m(A)$ is close to

$$J g_{f(a)}(a_{k+1}, \cdots, a_m) JF(a) = Jf(a).$$

PART 6. *If* $a \in X$ *and* f *is continuously differentiable in a neighborhood of* a, *then* $\mu'(a) = Jf(a)$.

Proof. Suppose $1<t<\infty$. Choose open neighborhoods U of a and V of $f(a)$, and continuously differentiable maps

$$P\colon U \to E_m \qquad \text{and} \qquad Q\colon V \to E_k$$

such that

$$t^{-1} \leqq \frac{|P(x) - P(z)|}{\text{distance }(x, z)} \leqq t \qquad \text{for } x, z \in U,$$
$$t^{-1} \leqq \frac{|Q(y) - Q(w)|}{\text{distance }(y, w)} \leqq t \qquad \text{for } y, w \in V.$$

It follows that suitable powers of t will serve as bounds for the effect of P and Q on Hausdorff measures and Jacobians. In fact suppose that $r>0$,

$$A = K(x, r) \subset U \qquad \text{and} \qquad f(A) \subset V,$$

let $F=Q\circ f\circ P^{-1}$ and observe that

$$A \cap f^{-1}\{y\} = P^{-1}[P(A) \cap F^{-1}\{Q(y)\}] \qquad \text{for } y \in V.$$

Applying Part 3 to f, and Parts 3 and 5 to F, one obtains

$$\begin{aligned}
\mu(A) &= \int_Y H^{m-k}(A \cap f^{-1}\{y\})dH^k y \\
&\leqq t^{m-k}\int_Y H^{m-k}[P(A) \cap F^{-1}\{Q(y)\}]dH^k y \\
&\leqq t^m \int_{E_k} H^{m-k}[P(A) \cap F^{-1}\{q\}]dH^k q \\
&= t^m \int_{P(A)} JF(p)dH^m p \\
&\leqq t^{m+2k}\int_{P(A)} Jf[P^{-1}(p)]dH^m p \\
&\leqq t^{2m+2k}\int_A Jf(x)dH^m x,
\end{aligned}$$

and similarly

$$\mu(A) \geqq t^{-2m-2k}\int_A Jf(x)dH^m x.$$

Dividing by $H^m(A)$ and letting $r\to 0+$, one concludes that the extreme limits of

$$\mu[K(a, r)]/H^m[K(a, r)]$$

lie between

$$t^{-2m-2k}Jf(a) \qquad \text{and} \qquad t^{2m+2k}Jf(a).$$

PART 7. *If A is an H^m measurable subset of X, then*

$$\mu(A) = \int_A Jf(x)dH^m x.$$

Proof. Proceeding as in [F2, 4.3] or [W1], choose disjoint closed subsets $C_1, C_2, C_3, \cdots$ of X and continuously differentiable maps $f_1, f_2, f_3, \cdots$ of X into Y such that

$$H^m\left(X - \bigcup_{i=1}^{\infty} C_i\right) = 0,$$

$$f \mid C_i = f_i \mid C_i \qquad \text{for } i = 1, 2, 3, \cdots.$$

Applying Parts 3 and 6 to f_i and observing that

$$Jf_i(x) = Jf(x)$$

whenever x is a point of density of C_i and f is differentiable at x, one obtains

$$\int_Y H^{m-k}(A \cap C_i \cap f^{-1}\{y\})dH^k y$$

$$= \int_Y H^{m-k}(A \cap C_i \cap f_i^{-1}\{y\})dH^k y$$

$$= \int_{A\cap C_i} Jf_i(x)dH^m x = \int_{A\cap C_i} Jf(x)dH^m x$$

for $i=1, 2, 3, \cdots$. Accordingly Part 1 implies that

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A \cap C_i)$$

$$= \sum_{i=1}^{\infty} \int_{A\cap C_i} Jf(x)dH^m x = \int_A Jf(x)dH^m x.$$

3.2. REMARK. The preceding argument shows also that $f^{-1}\{y\}$ is countably Hausdorff $m-k$ rectifiable (see [F4]) for H^k almost all y in Y.

In case X is a submanifold of class 1 of E_n, $H^{m-k}(A\cap f^{-1}\{y\})$ may be computed for H^k almost all y in Y by means of the integralgeometric formula [F4, 5.14], and one obtains

$$\int_Y H^{m-k}(A \cap f^{-1}\{y\})dH^k y = \beta(n, m - k)^{-1} \int_{G_n \times E_{m-k}} u(R, w)d(\phi_n \otimes L_{m-k})(R, w),$$

where

$$u(R, w) = \int_Y H^0[A \cap \lambda_n^{n-m+k}(R, w) \cap f^{-1}\{y\}]dH^k y$$

is the classical area of $f| [A \cap \lambda_n^{n-m+k}(R, w)]$. It follows that, if A is open in X, then the above integrals depend lowersemicontinuously on f, and undoubtedly it would be possible to develop (for $m \geqq k$) a theory of "coarea" dual to the existing (for $m \leqq k$) theory of Lebesgue area [R; CE; F8; DF].

4. **Sets with positive reach.** Here these sets are introduced, and are shown to have quite reasonable metric and tangential properties. If two sets in suitably general relative position belong to this class, so does their intersection. The class contains all convex sets, as well as all those sets which can be defined locally by means of finitely many equations, $f(x) = 0$, and inequalities, $f(x) \leqq 0$, using real valued continuously differentiable functions, f, whose gradients are Lipschitzian and satisfy a certain independence condition; therefore regular submanifolds of class 2 of E_n, with or without regular boundary, are included.

The concept of reach originates from the unique nearest point property, but toward the end of this section it is proved that a closed set has positive reach if and only if it makes uniform second order contact with its tangent cones. Then it follows that the class of sets with positive reach is closed under bi-Lipschitzian maps with Lipschitzian differentials.

4.1. DEFINITION. If $A \subset E_n$, then δ_A is the function on E_n such that

$$\delta_A(x) = \text{distance}(x, A) = \inf \{ |x - a| : a \in A\}$$

whenever $x \in E_n$. Furthermore

$$\text{Unp}(A)$$

is the set of all those points $x \in E_n$ for which there exists a unique point of A nearest to x, and the map

$$\xi_A : \text{Unp}(A) \to A$$

associates with $x \in \text{Unp}(A)$ the unique $a \in A$ such that $\delta_A(x) = |x - a|$.

If $a \in A$, then

$$\text{reach}(A, a)$$

is the supremum of the set of all numbers r for which

$$\{x : |x - a| < r\} \subset \text{Unp}(A).$$

Also

$$\text{reach}(A) = \inf \{\text{reach}(A, a) : a \in A\}.$$

4.2. REMARK. Suppose $A \subset E_n$. Then reach(A, a) is continuous with respect to $a \in A$, and

$$0 \leqq \text{reach}(\text{Boundary } A, a) \leqq \text{reach}(A, a) \leqq \infty$$

for $a \in \text{Boundary } A$. If $\text{reach}(A) > 0$, then A is closed.

A well known characterization of convexity, deducible from 4.8(8), states that $\text{reach}(A) = \infty$ if and only if A is convex and closed.

4.3. DEFINITION. If $A \subset E_n$ and $a \in A$, then the set

$$\text{Tan}(A, a)$$

of all *tangent vectors of A at a* consists of all those $u \in E_n$ such that either u is the null vector or for every $\epsilon > 0$ there exists a point $b \in A$ with

$$0 < |b - a| < \epsilon \quad \text{and} \quad \left| \frac{b - a}{|b - a|} - \frac{u}{|u|} \right| < \epsilon.$$

4.4. DEFINITION. If $A \subset E_n$ and $a \in A$, then the set

$$\text{Nor}(A, a)$$

of all *normal vectors of A at a* consists of all those $v \in E_n$ such that

$$v \bullet u \leqq 0 \quad \text{whenever} \quad u \in \text{Tan}(A, a).$$

4.5. REMARK. Recall that a subset C of E_n is a *convex* cone if and only if $x + y \in C$ and $\lambda x \in C$ whenever $x, y \in C$ and $\lambda > 0$. For every subset S of E_n,

$$\text{Dual}(S) = \{v : v \bullet u \leqq 0 \text{ for all } u \in S\}$$

is a closed convex cone, and $\text{Dual}[\text{Dual}(S)]$ is the smallest nonempty closed convex cone containing S. Furthermore

$$\text{Dual}(S_1 + S_2) = \text{Dual}(S_1) \cap \text{Dual}(S_2)$$

for any two subsets S_1 and S_2 of E_n containing the origin, and

$$\text{Dual}(S_1 \cap S_2) = \text{Dual}(S_1) + \text{Dual}(S_2)$$

in case S_1 and S_2 are closed convex cones. Also, for every closed convex cone C,

$$\dim C + \dim \text{Dual}(C) \geqq n$$

with equality holding if and only if C is a vectorspace.

Accordingly

$$\text{Nor}(A, a) = \text{Dual}[\text{Tan}(A, a)]$$

is always a closed convex cone, while $\text{Tan}(A, a)$ is closed and positively homogeneous but not necessarily convex.

4.6. REMARK. If A is a submanifold of class 1 of E_n, $f: A \to E_n$ is the inclusion map, and $a \in A$, then df maps the intrinsic tangent space of A at a isometrically onto $\text{Tan}(A, a)$.

4.7. LEMMA. *Suppose f is a real valued Lipschitzian function on an open* *subset W of E_n, j is an integer between* 1 *and n, and g is a real valued continuous function on W such that*

$$D_j f(x) = g(x) \text{ whenever } f \text{ is differentiable at } x.$$

Then

$$D_j f(x) = g(x) \qquad \text{for all } x \in W.$$

Proof. Suppose $w \in W$, $r > 0$ and $\{x: |x-w| < 2r\} \subset W$. Let y be the jth unit vector. According to Rademacher's theorem f is differentiable L_n almost everywhere in W, and for L_n almost all x within r of w it is true that

$$f(x + ty) - f(x) = \int_0^t D_j f(x + uy) du = \int_0^t g(x + uy) du$$

whenever $|t| < r$. From the continuity of f and g it follows that

$$f(w + ty) - f(w) = \int_0^t g(w + uy) du \quad \text{whenever} \quad |t| < r,$$

and finally that $D_j f(w) = g(w)$.

4.8. THEOREM. *For every nonempty closed subset A of E_n the following statements hold, with $\delta = \delta_A$, $\xi = \xi_A$, $U = \mathrm{Unp}(A)$:*

(1) $|\delta(x) - \delta(y)| \leqq |x - y|$ *whenever $x, y \in E_n$.*

(2) *If $a \in A$ and*

$$P = \{v: \xi(a + v) = a\}, \qquad Q = \{v: \delta(a + v) = |v|\},$$

then P and Q are convex and $P \subset Q \subset \mathrm{Nor}(A, a)$.

(3) *If $x \in E_n - A$ and δ is differentiable at x, then $x \in U$ and*

$$\mathrm{grad}\, \delta(x) = \frac{x - \xi(x)}{\delta(x)}.$$

(4) *ξ is continuous.*

(5) *δ is continuously differentiable on* $\mathrm{Int}(U - A)$ *and δ^2 is continuously differentiable on* Int U *with*

$$\mathrm{grad}\, \delta^2(x) = 2[x - \xi(x)] \qquad \text{for } x \in \mathrm{Int}\, U.$$

(6) *If $a \in A$, $v \in E_n$ and*

$$0 < \tau = \sup\{t: \xi(a + tv) = a\} < \infty,$$

then $a + \tau v \notin \mathrm{Int}\, U$.

(7) *If $x \in U$, $a = \xi(x)$,* reach$(A, a) > 0$ *and $b \in A$, then*

$$(x - a) \bullet (a - b) \geqq - \frac{|a - b|^2 |x - a|}{2\, \mathrm{reach}(A, a)}.$$

(8) *If* $0<r<q<\infty$, $x\in U$, $y\in U$ *and*

$$\delta(x) \leqq r, \qquad \delta(y) \leqq r, \qquad \operatorname{reach}[A, \xi(x)] \geqq q, \qquad \operatorname{reach}[A, \xi(y)] \geqq q,$$

then

$$|\xi(x) - \xi(y)| \leqq \frac{q}{q-r} |x - y|.$$

(9) *If* $0<s<r<\operatorname{reach}(A)$, *then* grad δ *is Lipschitzian on* $\{x: s\leqq\delta(x)\leqq r\}$, *and* grad δ^2 *is Lipschitzian on* $\{x: \delta(x)\leqq r\}$.

(10) *If* $a\in A$, *then*

$$\operatorname{Tan}(A, a) = \left\{u: \liminf_{t\to 0+} t^{-1}\delta(a + tu) = 0\right\}.$$

(11) *If* $a\in A$, $\operatorname{reach}(A, a)>r>0$, $u\in E_n$ *and*

$$u \bullet v \leqq 0 \quad \textit{whenever} \quad \xi(a+v) = a, \ |v| = r,$$

then

$$\lim_{t\to 0+} t^{-1}\delta(a + tu) = 0.$$

(12) *If* $a\in A$ *and* $\operatorname{reach}(A, a)>r>0$, *then*

$$\operatorname{Nor}(A, a) = \{\lambda v: \lambda \geqq 0, \ |v| = r, \xi(a+v) = a\},$$

$\operatorname{Tan}(A, a)$ *is the convex cone dual to* $\operatorname{Nor}(A, a)$, *and*

$$\lim_{t\to 0+} t^{-1}\delta(a + tu) = 0 \qquad \textit{for } u \in \operatorname{Tan}(A, a).$$

(13) *If*

$$N = \{(a, v): a \in A \ \textit{ and } \ v \in \operatorname{Nor}(A, a)\},$$
$$\sigma: N \to E_n, \ \sigma(a, v) = a + v \textit{ for } (a, v) \in N,$$
$$\psi: U \to E_n \times E_n, \ \psi(x) = (\xi(x), x - \xi(x)) \textit{ for } x \in U,$$

then

$$\sigma(N) = E_n, \qquad \sigma \textit{ is Lipschitizian},$$
$$\psi(U) \subset N, \qquad \psi \textit{ is a homeomorphism}, \qquad \psi^{-1} = \sigma | \psi(U).$$

If furthermore

$$K \subset A, 0 < r < q, \operatorname{reach}(A, a) \geqq q \textit{ for } a \in K,$$
$$W = U \cap \{x: \xi(x) \in K \ \textit{ and } \ \delta(x) \leqq r\},$$

then

$$\psi(W) = N \cap \{(a, v): a \in K \text{ and } |v| \leqq r\},$$
$$\psi | W \text{ is Lipschitzian;}$$

in case K is compact and $0 \leqq t < \infty$, *then*

$$N \cap \{(a, v): a \in K \text{ and } |v| \leqq t\} \text{ is compact.}$$

Proof of (1). Choosing $a \in A$ so that $\delta(x) = |x-a|$, one obtains

$$\delta(y) - \delta(x) \leqq |y - a| - |x - a| \leqq |y - x|.$$

Proof of (2). Assume $a = 0$ and note that

$$v \in P \text{ if and only if } |b - v| > |v| \text{ for all } b \in A - \{a\},$$
$$v \in Q \text{ if and only if } |b - v| \geqq |v| \text{ for all } b \in A.$$

Furthermore

$$|b - v|^2 - |v|^2 = b \bullet (b - 2v) \quad \text{whenever} \quad b, v \in E_n,$$

and consequently, if $b, v, w \in E_n$, $s \geqq 0$, $t \geqq 0$, $s + t = 1$, then

$$\begin{aligned} |b - (sv + tw)|^2 - |sv + tw|^2 &= b \bullet (b - 2sv - 2tw) \\ &= b \bullet [s(b - 2v) + t(b - 2w)] = sb \bullet (b - 2v) + tb \bullet (b - 2w) \\ &= s(|b - v|^2 - |v|^2) + t(|b - w|^2 - |w|^2). \end{aligned}$$

It follows that P and Q are convex, and clearly $P \subset Q$.

Finally suppose $v \in Q$. If $b \in A - \{a\}$, then $|b|^2 \geqq 2b \bullet v$, hence

$$v \bullet \frac{b}{|b|} \leqq \frac{|b|}{2}.$$

This shows that $v \bullet u \leqq 0$ for $u \in \operatorname{Tan}(A, a)$.

Proof of (3). If $a \in A$ and $\delta(x) = |x - a|$, then (2) implies

$$\delta[x + t(a - x)] = \delta(x) - t\delta(x) \qquad \text{for } 0 \leqq t \leqq 1,$$

whence

$$\operatorname{grad} \delta(x) \bullet \frac{x - a}{\delta(x)} = \frac{D\delta(x)(a - x)}{-\delta(x)} = 1.$$

Since $|\operatorname{grad} \delta(x)| \leqq 1$, by (1), it follows that $\operatorname{grad} \delta(x) = (x - a)/\delta(x)$.

Proof of (4). Otherwise there exists an $\epsilon > 0$ and a sequence $x_1, x_2, x_3, \cdots$ of points of U convergent to a point $x \in U$ and such that $|\xi(x_i) - \xi(x)| \geqq \epsilon$ for $i = 1, 2, 3, \cdots$. Then

$$|\xi(x_i) - x_i| = \delta(x_i), \qquad |\xi(x_i) - x| \leqq \delta(x) + 2|x_i - x|,$$

hence all the points $\xi(x_i)$ lie in a bounded subset of the closed set A, and passing to a subsequence one may assume that the sequence $\xi(x_1), \xi(x_2), \xi(x_3), \cdots$

converges to a point $a \in A$. But then

$$\delta(x) = \lim_{i\to\infty} \delta(x_i) = \lim_{i\to\infty} | \xi(x_i) - x_i | = | a - x | ,$$

hence $a = \xi(x)$, which is incompatible with

$$| a - \xi(x) | = \lim_{i\to\infty} | \xi(x_i) - \xi(x) | \geqq \epsilon.$$

Proof of (5). According to (1) and (4) the right member of the equation in (3) represents a continuous map of $U - A$ into E_n. Since the components of the left member of this equation are $D_1\delta(x), \cdots, D_n\delta(x)$, it follows from Lemma 4.7 that δ has continuous partial derivatives on $W = \operatorname{Int}(U-A)$.

In case $x \in W$, the stated formula for grad $\delta^2(x)$ follows from the equation in (3). In case $x \in A$, $\delta^2(x+h) \leqq |h|^2$ for $h \in E_n$, hence grad $\delta^2(x) = 0$, and also $\xi(x) = x$. Accordingly the formula holds for all $x \in \operatorname{Int} U$, and the continuity of the right member, guaranteed by (4), implies the continuity of grad δ^2 on Int U.

Proof of (6). Assume $|v| = 1$ and $y = a + \tau v \in \operatorname{Int} U$. Then (4) and (3) imply that $\xi(y) = a$, $\delta(y) = \tau$, $y \notin A$, grad $\delta(y) = v$.

In view of (5) one may apply Peano's existence theorem for solutions of differential equations to obtain an $r > 0$ and a map

$$C: \{s: -r < s < r\} \to \operatorname{Int}(U - A)$$

such that

$$C' = (\operatorname{grad} \delta) \circ C \quad \text{and} \quad C(0) = y.$$

If $|s| < r$, then $|C'(s)| = |\operatorname{grad} \delta[C(s)]| = 1$ and

$$(\delta \circ C)'(s) = \operatorname{grad} \delta[C(s)] \bullet C'(s) = C'(s) \bullet C'(s) = 1.$$

Accordingly, if $-r < p < q < r$, then

$$\int_p^q | C'(s) | \, ds = \int_p^q (\delta \circ C)'(s) ds = \delta[C(q)] - \delta[C(p)] \leqq | C(q) - C(p) | .$$

It follows that the curve C parameterizes a straight line segment in the direction $C'(0) = \operatorname{grad} \delta(y) = v$.

If $0 < s < r$ and $t = \tau + s$, then

$$C(s) = y + sv = a + tv, \qquad \delta[C(s)] = \delta(y) + s = t = | C(s) - a | ,$$

hence $\xi(a + tv) = a$, with $t > \tau$, contrary to the definition of τ.

Proof of (7). Assume $x \neq a$ and let

$$v = \frac{x - a}{| x - a |}, \qquad S = \{t: \xi(a + tv) = a\}.$$

Since $|x-a|\in S$, $\sup S>0$ and it follows from (6) that

$$\sup S \geqq \operatorname{reach}(A, a).$$

Moreover, if $0<t\in S$, then

$$|a+tv-b| \geqq \delta(a+tv) = t, \qquad |a-b|^2 + 2tv \bullet (a-b) + t^2 \geqq t^2,$$

$$2tv \bullet (a-b) \geqq -|a-b|^2, \quad (x-a) \bullet (a-b) \geqq -|a-b|^2|x-a|/2t.$$

Proof of (8). Letting $a=\xi(x)$ and $b=\xi(y)$, one infers from (7) that

$$(x-a) \bullet (a-b) \geqq -|a-b|^2 r/2q$$

and symmetrically

$$(y-b) \bullet (b-a) \geqq -|b-a|^2 r/2q.$$

Therefore

$$\begin{aligned} |x-y| \cdot |a-b| &\geqq (x-y) \bullet (a-b) \\ &= [(a-b)+(x-a)+(b-y)] \bullet (a-b) \\ &\geqq |a-b|^2(1-r/q), \\ |x-y| &\geqq |a-b|(q-r)/q. \end{aligned}$$

Proof of (9). Combine (8), (1), (3) and (5).

Proof of (10). Suppose $a=0$ and $|u|=1$.

If $u\in T(A, a)$ and $\epsilon>0$, then there exists a point $b\in A$ such that

$$0 < |b| < \epsilon \quad \text{and} \quad \left|\frac{b}{|b|} - u\right| < \epsilon,$$

hence

$$|b|^{-1}\delta(|b|u) \leqq |b|^{-1}||b|u-b| = \left|u - \frac{b}{|b|}\right| < \epsilon.$$

On the other hand, suppose that whenever $0<\epsilon<1$ there exists a number t such that

$$0 < t < \epsilon \quad \text{and} \quad t^{-1}\delta(tu) < \epsilon;$$

choosing $b\in A$ so that $\delta(tu)=|tu-b|$, one finds that

$$|t-|b|| \leqq \delta(tu) < \epsilon t, \qquad 0 < (1-\epsilon)t < |b| < (1+\epsilon)t < \epsilon + \epsilon^2,$$

$$\left|\frac{b}{|b|} - u\right| = \frac{|b-|b|u|}{|b|} \leqq \frac{|b-tu| + |t-|b||}{|b|} \leqq \frac{2\epsilon t}{(1-\epsilon)t} = \frac{2\epsilon}{1-\epsilon}.$$

Proof of (11). Suppose $a=0$, $|u|=1$ and

$$\limsup_{t\to 0+} t^{-1}\delta(tu) > 0.$$

Choose ϵ and S so that

$$0 < \epsilon < \text{reach}(A, a) - r, \qquad S \subset \{t: 0 < t < \epsilon\},$$
$$0 \in \text{Closure } S, \qquad \delta(tu) > t\epsilon \text{ for } t \in S.$$

If $t \in S$, then

$$\delta(tu) \leqq |tu| = t < \epsilon < \text{reach}(A, a), \qquad tu \in U.$$

For $t \in S$, $0 \leqq \rho \leqq r$ let

$$\eta(t, \rho) = tu + \rho \text{ grad } \delta(tu) = \xi(tu) + [\delta(tu) + \rho] \text{ grad } \delta(tu)$$

and observe that

$$|\eta(t, \rho)| \leqq \epsilon + r < \text{reach}(A, a), \qquad \eta(t, \rho) \in \text{Int } U.$$

It follows from (6), with a and v replaced by $\xi(tu)$ and grad $\delta(tu)$, that

$$\xi[\eta(t, r)] = \xi(tu) \quad \text{whenever} \quad t \in S.$$

Inasmuch as $\{\eta(t, r): t \in S\}$ is bounded, one may assume, after replacing S by a suitable subset, that there exists a point $v \in E_n$ for which

$$\lim_{S \ni t \to 0} \eta(t, r) = v.$$

Then

$$|v| = \lim_{S \ni t \to 0} |\eta(t, r)| = r, \qquad v \in U,$$
$$\xi(v) = \lim_{S \ni t \to 0} \xi[\eta(t, r)] = a,$$

and consequently, by hypothesis, $u \bullet v \leqq 0$.

Choosing $t \in S$ so that

$$u \bullet \eta(t, r) < \epsilon r,$$

one may use the fact that

$$\delta[\eta(t, r)] \leqq |\eta(t, r)|$$

to obtain

$$[\delta(tu) + r]^2 \leqq |tu + [\eta(t, r) - tu]|^2,$$
$$[\delta(tu)]^2 + 2r\delta(tu) + r^2 \leqq t^2 + 2tu \bullet [\eta(t, r) - tu] + r^2,$$
$$2r\delta(tu) < t^2 + 2t\epsilon r - 2t^2 < 2t\epsilon r,$$

hence $\delta(tu) < t\epsilon$, contrary to the choice of ϵ and S.

Proof of (12). Since $\{a+v: |v| \leqq r\} \subset U$, the set

$$S = \{\lambda v: \lambda \geqq 0, |v| = r, \xi(a + v) = a\}$$

is closed. Clearly S is positively homogeneous. In order to verify that S is additive, suppose

$$\lambda > 0, \quad |v| = r, \quad \xi(a+v) = a, \quad \mu > 0, \quad |w| = r, \quad \xi(a+w) = a,$$

let

$$z = (\lambda + \mu)^{-1}(\lambda v + \mu w),$$

and use (2) and (6) to infer that

$$\xi(a+z) = a, \qquad \xi(a + r|z|^{-1}z) = a,$$
$$\lambda v + \mu w = (|\lambda v + \mu w| r^{-1})(r|z|^{-1}z) \in S.$$

Thus S is a closed convex cone.

Now let

$$L = \left\{u: \lim_{t\to 0+} t^{-1}\delta(a+tu) = 0\right\}.$$

One sees from 4.5 that

$$\mathrm{Tan}(A, a) \subset \mathrm{Dual}[\mathrm{Nor}(A, a)],$$

from (2) that

$$S \subset \mathrm{Nor}(A, a), \quad \text{hence} \quad \mathrm{Dual}[\mathrm{Nor}(A, a)] \subset \mathrm{Dual}(S),$$

and from (11) and (10) that

$$\mathrm{Dual}(S) \subset L \subset \mathrm{Tan}(A, a).$$

Accordingly

$$\mathrm{Tan}(A, a) = \mathrm{Dual}[\mathrm{Nor}(A, a)] = \mathrm{Dual}(S) = L, \qquad \mathrm{Nor}(A, a) = S.$$

Proof of (13). One sees from (2) that if $x \in E_n$, $a \in A$ and $\delta(x) = |x-a|$, then

$$x - a \in \mathrm{Nor}(A, a), \qquad (a, x-a) \in N, \qquad \sigma(a, x-a) = x.$$

In case $x \in U$, then $a = \xi(x)$, $\psi(x) = (a, x-a)$, $\sigma[\psi(x)] = x$. This implies the first part of (13). The second part follows from (12), (2), and (8); in case K is compact, so are W, $\psi(W)$ and the image of $\psi(W)$ under the transformation mapping (a, v) onto $(a, tr^{-1}v)$.

4.9. Corollary. *If* $s > 0$ *and* $A_s = \{x: \delta_A(x) \leqq s\}$, *then*

$$\delta_{A_s}(x) = \delta_A(x) - s \quad \textit{whenever} \quad \delta_A(x) \geqq s,$$
$$\xi_A[\xi_{A_s}(x)] = \xi_A(x) \quad \textit{whenever} \quad \delta_A(x) < \mathrm{reach}(A),$$
$$\mathrm{reach}(A_s) \geqq \mathrm{reach}(A) - s.$$

Furthermore, if $0 < s < \mathrm{reach}(A)$ *and* $A'_s = \{x: \delta_A(x) \geqq s\}$, *then*

$$\delta_{A'_s}(x) = s - \delta_A(x) \quad \textit{whenever} \quad 0 < \delta_A(x) \leqq s,$$
$$\xi_A[\xi_{A'_s}(x)] = \xi_A(x) \quad \textit{whenever} \quad 0 < \delta_A(x) \leqq s,$$
$$\text{reach}(A'_s) \geqq s.$$

Proof. The formula for δ_{A_s} follows mechanically from the definitions, and the formula for $\delta_{A'_s}$ may be derived with the aid of 4.8 (6). Then the statements concerning reach and ξ can be obtained from 4.8 (5) and (3), applied to A, A_s and A'_s.

4.10. THEOREM. *Suppose*

$$A \textit{ and } B \textit{ are closed subsets of } E_n,$$
$$C \textit{ is a nonempty compact subset of } A \cap B,\ r > 0,$$
$$\text{reach}(A, c) > r \textit{ and } \text{reach}(B, c) > r \quad \textit{for } c \in C,$$

and there exist no c and v such that

$$c \in C, \qquad v \in \text{Nor}(A, c), \qquad -v \in \text{Nor}(B, c), \qquad v \neq 0.$$

Let η be the infimum of the set consisting of 1 *and the numbers*

$$\frac{|v + w|}{|v| + |w|}$$

corresponding to $v \in \text{Nor}(A, c)$, $w \in \text{Nor}(B, c)$, $c \in C$ *with* $|v| + |w| > 0$. *Then*:

(1) $0 < \eta \leqq 1$ *and there exists a* ζ *such that* $0 < \zeta \leqq r$ *and*

$$|\lambda \text{ grad } \delta_A(x) + \mu \text{ grad } \delta_B(x)| > (\eta/2)(\lambda + \mu)$$

whenever $x \in E_n - (A \cup B)$, $\delta_C(x) < \zeta$, $\lambda > 0$, $\mu > 0$.

(2) $\delta_{A \cap B}(x) \leqq (2/\eta)[\delta_A(x) + \delta_B(x)]$ *whenever* $\delta_C(x) < \eta\zeta/5$.

(3) *If* $c \in C$, *then*

$$\text{Tan}(A \cap B, c) = \text{Tan}(A, c) \cap \text{Tan}(B, c),$$
$$\text{Nor}(A \cap B, c) = \text{Nor}(A, c) + \text{Nor}(B, c).$$

(4) *If* $c \in A \cap B$, $0 < \rho \leqq r\eta/2$ *and*

$$A \cap B \cap \{z : |z - c| < 2\rho\} \subset C,$$

then $\text{reach}(A \cap B, c) \geqq \rho$.

(5) *If* $C = A \cap B$, *then* $\text{reach}(A \cap B) \geqq r\eta/2$.

Proof of (1). For $0 \leqq \epsilon \leqq r$ let

$$S(\epsilon) = A \cap \{a : \delta_C(a) \leqq \epsilon\}, \qquad T(\epsilon) = B \cap \{b : \delta_C(b) \leqq \epsilon\},$$
$$M(\epsilon) = \{(a, v) : a \in S(\epsilon),\ v \in \text{Nor}(A, a),\ |v| \leqq 1\},$$
$$N(\epsilon) = \{(b, w) : b \in T(\epsilon),\ w \in \text{Nor}(B, b),\ |w| \leq 1\},$$
$$P(\epsilon) = [M(\epsilon) \times N(\epsilon)] \cap \{((a, v), (b, w)) : |a - b| \leqq \epsilon,\ |v| + |w| = 1\},$$

observe that $S(\epsilon)$ and $T(\epsilon)$ are compact with

$$\operatorname{reach}(A, a) > r - \epsilon \qquad \text{for } a \in S(\epsilon),$$
$$\operatorname{reach}(B, b) > r - \epsilon \qquad \text{for } b \in T(\epsilon),$$

and use 4.8 (13) to infer that $M(\epsilon)$, $N(\epsilon)$ and $P(\epsilon)$ are compact. Furthermore let Δ be the function on $P(r)$ such that

$$\Delta((a, v), (b, w)) = |v + w| \qquad \text{for } ((a, v), (b, w)) \in P(r),$$

and note that Δ is a continuous function, Δ does not vanish on $P(0)$, and either $P(0)$ is empty or η is the minimum value of Δ on $P(0)$, hence $0 < \eta \leqq 1$. Moreover, since

$$P(0) = \bigcap_{0<\epsilon<r} P(\epsilon),$$

one may choose ϵ so that $0 < \epsilon < r$ and the minimum value of Δ on $P(\epsilon)$ exceeds $\eta/2$.

Let $\zeta = \epsilon/2$ and suppose

$$x \in E_n - (A \cup B), \qquad \delta_C(x) < \zeta, \qquad \lambda > 0, \qquad \mu > 0.$$

Choosing a, b, v, w so that

$$a \in A, \quad \delta_A(x) = |x - a|, \quad b \in B, \quad \delta_B(x) = |x - b|,$$
$$(\lambda + \mu)v = \lambda \operatorname{grad} \delta_A(x), \quad (\lambda + \mu)w = \mu \operatorname{grad} \delta_B(x),$$

one readily verifies with the help of 4.8 (2) that

$$((a, v), (b, w)) \in P(\epsilon), \quad \text{hence} \quad |v + w| > \eta/2.$$

Proof of (2). Letting

$$\psi = [(\delta_A)^2 + (\delta_B)^2]^{1/2},$$
$$Q = \{x: \delta_C(x) < \zeta\} - (A \cap B),$$

one sees from 4.8 (5) that ψ is continuously differentiable on Q. Furthermore

$$|\operatorname{grad} \psi(x)| \geqq \eta/2 \qquad \text{for } x \in Q.$$

In fact, if $x \in Q - (A \cup B)$, then

$$\operatorname{grad} \psi(x) = [\delta_A(x) \operatorname{grad} \delta_A(x) + \delta_B(x) \operatorname{grad} \delta_B(x)]/\psi(x),$$
$$|\operatorname{grad} \psi(x)| \geqq (\eta/2)[\delta_A(x) + \delta_B(x)]/\psi(x) \geqq \eta/2$$

by virtue of (1); on the other hand

$$\operatorname{grad} \psi(x) = \operatorname{grad} \delta_B(x) \qquad \text{for } x \in Q \cap A,$$
$$\operatorname{grad} \psi(x) = \operatorname{grad} \delta_A(x) \qquad \text{for } x \in Q \cap B,$$

hence $|\operatorname{grad} \psi(x)| = 1$ for $x \in Q \cap (A \cup B)$.

Fix a point $z \in Q$ such that $\delta_C(z) < \eta\zeta/5$ and consider the class of all maps

$$q: J \to Q$$

such that J is an open real interval containing 0,

$$q(0) = z \quad \text{and} \quad q' = -(\operatorname{grad} \psi) \circ q.$$

Since this class is nonempty, according to Peano's existence theorem for solutions of differential equations, and is inductively ordered by extension, it has a maximal element. Henceforth let $q: J \to Q$ be such a maximal element.

If $t \in J$, then $q'(t) = -\operatorname{grad} \psi[q(t)]$, hence

$$|q'(t)| \geqq \eta/2, \quad (\psi \circ q)'(t) = \operatorname{grad} \psi[q(t)] \bullet q'(t) = -|q'(t)|^2.$$

It follows that if $0 < u \in J$, then

$$\begin{aligned} \psi(z) = \psi[q(0)] &\geqq \psi[q(0)] - \psi[q(u)] \\ &= \int_0^u |q'(t)|^2 dt \geqq (\eta/2) \int_0^u |q'(t)| \, dt \geqq u\eta^2/4. \end{aligned}$$

Consequently $\tau = \sup J < \infty$ and there exists a point $h \in E_n$ such that

$$\lim_{t \to \tau-} q(t) = h,$$

$$|h - z| \leqq \int_0^\tau |q'(t)| \, dt = (2/\eta)\psi(z) \leqq (2/\eta)[\delta_A(z) + \delta_B(z)].$$

The proof will be completed by showing that $h \in A \cap B$. Otherwise, since

$$\delta_C(h) \leqq |h - z| + \delta_C(z) \leqq [(4/\eta) + 1]\delta_C(z) < \zeta,$$

it would be true that $h \in Q$, and Peano's existence theorem would furnish an $\epsilon > 0$ and a map

$$p: \{t: \tau - \epsilon < t < \tau + \epsilon\} \to Q$$

for which $p(\tau) = h$ and $p' = -(\operatorname{grad} \psi) \circ p$. Inasmuch as

$$\lim_{t \to \tau-} q'(t) = \lim_{t \to \tau-} -\operatorname{grad} \psi[q(t)] = -\operatorname{grad} \psi(h) = p'(\tau),$$

the map

$$\begin{aligned} &P: J \cup \{t: \tau \leqq t < \tau + \epsilon\} \to Q, \\ &P(t) = q(t) \text{ for } t \in J, \qquad P(t) = p(t) \text{ for } \tau \leqq t < \tau + \epsilon, \end{aligned}$$

would be a proper extension of $q: J \to Q$ with

$$P(0) = z \quad \text{and} \quad P' = -(\operatorname{grad} \psi) \circ P,$$

contrary to the maximal property of q.

Proof of (3). Inasmuch as

$$\delta_{A\cap B}(x) \geqq \delta_A(x) \quad \text{and} \quad \delta_{A\cap B}(x) \geqq \delta_B(x) \qquad \text{for } x \in E_n,$$

threefold application of 4.8 (10) yields

$$\operatorname{Tan}(A \cap B, c) \subset \operatorname{Tan}(A, c) \cap \operatorname{Tan}(B, c).$$

On the other hand, if $u\in\operatorname{Tan}(A, c)\cap\operatorname{Tan}(B, c)$, one infers from 4.8 (12) that

$$\lim_{t\to 0+} t^{-1}\delta_A(c + tu) = 0 \quad \text{and} \quad \lim_{t\to 0+} t^{-1}\delta_B(c + tu) = 0,$$

hence from (2) and 4.8 (10) that

$$\lim_{t\to 0+} t^{-1}\delta_{A\cap B}(c + tu) = 0, \qquad u \in \operatorname{Tan}(A \cap B, c).$$

This proves the first equation in (3), and the second now follows from 4.5 and 4.8 (12).

Proof of (4). Suppose $|x-c| <\rho$, $z\in A\cap B$, $\delta_{A\cap B}(x)=|x-z|$.

One sees from 4.8 (2) that $x-z\in\operatorname{Nor}(A\cap B, z)$. Inasmuch as

$$|z - c| \leqq |z - x| + |x - c| \leqq \delta_{A\cap B}(x) + |x - c| \leqq 2|x - c| < 2\rho,$$

hence $z\in C$, it follows from (3) that there exist v and w with

$$v \in \operatorname{Nor}(A, z), \qquad w \in \operatorname{Nor}(B, z), \qquad v + w = x - z.$$

Now $\eta(|v|+|w|)\leqq|v+w|=|x-z|<\rho\leqq r\eta/2$, hence

$$|2v| \leqq r \quad \text{and} \quad |2w| \leqq r.$$

Since $\operatorname{reach}(A, z)>r$ and $\operatorname{reach}(B, z)>r$, one infers from 4.8 (12) that

$$\xi_A(z + 2v) = z \quad \text{and} \quad \xi_B(z + 2w) = z.$$

Recalling that $z\in A\cap B$ one obtains

$$\xi_{A\cap B}(z + 2v) = z \quad \text{and} \quad \xi_{A\cap B}(z + 2w) = z,$$

and one concludes from 4.8 (2) that

$$z = \xi_{A\cap B}[z + (2v + 2w)/2] = \xi_{A\cap B}(x).$$

Proof of (5). Applying (4) to all $c\in C$ with $\rho=r\eta/2$.

4.11. Lemma. *Suppose f is a continuously differentiable real valued function on an open subset of E_n,* grad f *is Lipschitzian, and*

$$A = \{x: f(x) = 0\}, \qquad B = \{x: f(x) \leqq 0\}.$$

If $a\in A$ and grad $f(a)\neq 0$, *then* $0<\operatorname{reach}(A, a)\leqq\operatorname{reach}(B, a)$.

Proof. Let M be a Lipschitzian constant for grad f, and choose positive numbers h and r such that

$$|\operatorname{grad} f(w)| \geqq h \quad \text{whenever} \quad |w - a| < r.$$

It will be shown that $\operatorname{reach}(A, a) \geqq s = \inf\{r/2, h/M\}$.

Suppose $|x-a| < s$, $b \in A$, $c \in A$, $|b-x| = |c-x| = \delta_A(x)$. Then $|b-a| < r$, $|c-a| < r$ and Taylor's Theorem implies that

$$|f(c) - f(b) - (c - b) \bullet \operatorname{grad} f(b)| \leqq |c - b|^2 M/2.$$

Furthermore $f(c) = f(b) = 0$, and since $x - b \in \operatorname{Nor}(A, b)$ according to 4.8 (2) there exists a real number t such that

$$x - b = t \operatorname{grad} f(b).$$

It follows that

$$|2(c - b) \bullet (b - x)| \leqq |c - b|^2 M |t|,$$

$$0 = |c - x|^2 - |b - x|^2 = |c - b|^2 + 2(c - b) \bullet (b - x)$$

$$\geqq |c - b|^2(1 - M|t|),$$

$$h/M > |x - b| = |t| \cdot |\operatorname{grad} f(b)| \geqq |t| h, 1 > M|t|,$$

hence $|c-b|^2 = 0$.

4.12. THEOREM. *Suppose $f_1, \cdots, f_m$ are continuously differentiable real valued functions on an open subset of E_n,* $\operatorname{grad} f_1, \cdots, \operatorname{grad} f_m$ *are Lipschitzian,* $0 \leqq k \leqq m$, *and*

$$A = \bigcap_{i=1}^{k} \{x : f_i(x) = 0\} \cap \bigcap_{i=k+1}^{m} \{x : f_i(x) \leqq 0\}.$$

If $a \in A$, $J = \{i : f_i(a) = 0\}$, and there do not exist real numbers t_i, corresponding to $i \in J$, such that $t_i \neq 0$ for some $i \in J$, $t_i \geqq 0$ whenever $i \in J$ and $i > k$,

$$\sum_{i \in J} t_i \operatorname{grad} f_i(a) = 0,$$

then $\operatorname{reach}(A, a) > 0$ *and*

$$\operatorname{Nor}(A, a) = \left\{ \sum_{i \in J} t_i \operatorname{grad} f_i(a) : t_i \geqq 0 \text{ whenever } i > k \right\}.$$

Proof. Using 4.11 and 4.10, apply induction with respect to m.

4.13. THEOREM. *Suppose $\epsilon > 0$. If $A_1, A_2, A_3, \cdots$ and B are closed subsets of E_n such that* $\operatorname{reach}(A_k) \geqq \epsilon$ *for $k = 1, 2, 3, \cdots$ and*

$$\delta_{A_k}(x) \to \delta_B(x) \text{ uniformly for } x \in C \text{ as } k \to \infty$$

whenever C is a compact subset of $\{x : \delta_B(x) < \epsilon\}$, then $\operatorname{reach}(B) \geqq \epsilon$ *and*

$$\xi_{A_k}(x) \to \xi_B(x) \text{ uniformly for } x \in C \text{ as } k \to \infty$$

whenever C is a compact subset of $\{x : \delta_B(x) < \epsilon\}$.

Proof. Suppose C is a compact subset of $\{x: \delta_B(x) < \epsilon\}$. Choose an open set W such that $C \subset W$ and the closure of W is a compact subset of $\{x: \delta_B(x) < \epsilon\}$, a number r such that

$$\sup \{\delta_B(x): x \in W\} < r < \epsilon,$$

and a positive integer K such that

$$\sup \{\delta_{A_k}(x): x \in W\} < r \qquad \text{for } k \geqq K.$$

It follows from 4.8 (8) that the functions $\xi_{A_k} | W$ corresponding to $k \geqq K$ have the common Lipschitz constant $\epsilon/(\epsilon - r)$, and hence from 4.8 (5) that the functions $(\delta_{A_k})^2$ are equiuniformly differentiable on W. Since

$$\delta^2_{A_k}(x) \to \delta^2_B(x) \text{ uniformly for } x \in W \text{ as } k \to \infty,$$

one infers that $(\delta_B)^2$ is uniformly differentiable on W and

$$\text{grad } \delta^2_{A_k}(x) \to \text{grad } \delta^2_B(x) \text{ uniformly for } x \in W \text{ as } k \to \infty.$$

Finally one uses 4.8 (3) and (5) to conclude that $W \subset \text{Unp}(B)$ and

$$\xi_{A_k}(x) \to \xi_B(x) \text{ uniformly for } x \in W \text{ as } k \to \infty.$$

4.14. Remark. Observing that if A and B are nonempty closed subsets of E_n, then

$$\sup_{x \in E_n} | \delta_A(x) - \delta_B(x) |$$

equals the Hausdorff distance between A and B, one sees from 4.13 that for each $\epsilon > 0$ the set

$$\{A: 0 \neq A \subset E_n \text{ and reach}(A) \geqq \epsilon\}$$

is closed with respect to the Hausdorff metric. It follows that if $\epsilon > 0$ and K is a compact subset of E_n, then

$$\{A: 0 \neq A \subset K \text{ and reach}(A) \geqq \epsilon\}$$

is compact.

4.15. Remark. The reasonable local behavior of a subset A of E_n, such that reach$(A) > 0$, is further illustrated by the following properties:

(1) *If* $p \in E_n$ *and* $0 < r < \text{reach}(A)$, *then*

$$A \cap \{x: | x - p | \leqq r\} \textit{ is contractible.}$$

(2) *If* $a \in A$, $\dim \text{Tan}(A, a) = k$ *and*

$$P(r) = A \cap \{x: | x - a | \leqq r\}, \qquad Q(r) = \text{Tan}(A, a) \cap \{u: | u | \leqq r\},$$

whenever $r > 0$, then $\dim (A) \geqq k$ *and*

$$\liminf_{r \to 0+} \frac{H^k[P(r)]}{H^k[Q(r)]} \geqq 1.$$

(3) *For* $k=0, 1, \cdots, n$ *the set*

$$A^{(k)} = A \cap \{a: \dim \operatorname{Nor}(A, a) \geqq n - k\}$$

is countably k *rectifiable.*

(4) *If* $\dim(A)=k$, *then* $A=A^{(k)} \neq A^{(k-1)}$ *and, for* $a \in A - A^{(k-1)}$, $\operatorname{Tan}(A, a)$ *is a* k *dimensional vectorspace.*

To prove (1), consider the homotopy h such that

$$h(x, t) = \xi_A[(1-t)x + tp]$$

whenever $x \in A$, $|x-p| \leqq r$, $0 \leqq t \leqq 1$.

To prove (2), assume $a=0$, let U be the k dimensional vectorspace containing $\operatorname{Tan}(A, a)$, and consider the continuous maps

$$f_t: Q(1) \to U, \qquad f_t(u) = t^{-1}(\xi_U \circ \xi_A)(tu) \text{ for } u \in Q(1),$$

corresponding to $0 < t < \operatorname{reach}(A)$. Inasmuch as

$$\begin{aligned} |f_t(u) - u| &= t^{-1}|\xi_U[\xi_A(tu) - tu]| \\ &\leqq t^{-1}|\xi_A(tu) - tu| = t^{-1}\delta_A(tu) \end{aligned}$$

for $u \in Q(1)$, and since one easily sees from 4.8 (12) that $t^{-1}\delta_A(tu) \to 0$ uniformly for $u \in Q(1)$ as $t \to 0+$, it follows that as $t \to 0+$ the maps f_t converge to the inclusion map of $Q(1)$ into U, whence $\dim(A) \geqq k$.

Given any ϵ such that $0 < \epsilon < 1$, one may choose $\rho > 0$ so that if $0 < t < \rho$ then

$$t^{-1}\delta_A(tu) < \epsilon \qquad \text{for } u \in Q(1),$$
$$H^k(f_t[Q(1)]) > (1-\epsilon)H^k[Q(1)].$$

One concludes that if $0 < r < \rho$ and $t = r(1+\epsilon)^{-1}$, then

$$\begin{aligned} \xi_A[Q(t)] &\subset P(r), \\ tf_t[Q(1)] &= (\xi_U \circ \xi_A)[Q(t)] \subset \xi_U[P(r)], \\ H^k[P(r)] \geqq t^k H^k(f_t[Q(1)]) &> (1+\epsilon)^{-k} r^k (1-\epsilon) H^k[Q(1)] \\ &= (1+\epsilon)^{-k}(1-\epsilon)H^k[Q(r)]. \end{aligned}$$

To prove (3), let S be a countable dense set of k dimensional planes in E_n, suppose $0 < r < \operatorname{reach}(A)$, observe that

$$A^{(k)} \subset \bigcup_{\sigma \in S} \xi_A[\sigma \cap \{x: \delta_A(x) \leqq r\}],$$

and recall 4.8 (8).

To prove (4), first use (2) to infer that if $a \in A$, then

$$\dim \operatorname{Tan}(A, a) \leqq \dim(A) \leqq k, \quad \text{hence} \quad a \in A^{(k)};$$

in case $a \notin A^{(k-1)}$, then

$$\dim \operatorname{Nor}(A, a) = n - k, \ \dim \operatorname{Tan}(A, a) \geqq k,$$
$$\dim \operatorname{Nor}(A, a) + \dim \operatorname{Tan}(A, a) = n,$$

hence $\operatorname{Tan}(A, a)$ is a k dimensional vectorspace. On the other hand (3) implies that $H^k[A^{(k-1)}] = 0$, hence $\dim A^{(k-1)} \leqq k-1$ according to [HW, VII], and consequently $A \not\subset A^{(k-1)}$.

4.16. LEMMA. *For every nonempty closed subset S of E_n,*

$$[\delta_S(x)]^2 + [\delta_{\operatorname{Dual}(S)}(x)]^2 \geqq |x|^2 \quad \textit{whenever} \quad x \in E_n;$$

furthermore S is a convex cone if and only if

$$[\delta_S(x)]^2 + [\delta_{\operatorname{Dual}(S)}(x)]^2 = |x|^2 \quad \textit{whenever} \quad x \in E_n.$$

Proof. If $x \in E_n$, $u \in S$, $\delta_S(x) = |x - u|$, then either $|x-u| \geqq |x|$ or $x \bullet u > 0$,

$$\operatorname{Dual}(S) \subset \operatorname{Dual}(\{u\}) = \{v : v \bullet u \leqq 0\},$$
$$\delta_{\operatorname{Dual}(S)}(x) \geqq \delta_{\operatorname{Dual}(\{u\})}(x) = (x \bullet u)/|u|,$$
$$[\delta_S(x)]^2 + [\delta_{\operatorname{Dual}(S)}(x)]^2 \geqq |x - u|^2 + [(x \bullet u)/|u|]^2$$
$$= |x|^2 + [|u| - (x \bullet u)/|u|]^2 \geqq |x|^2;$$

in case S is a convex cone it is also true that

$$S \subset \operatorname{Tan}(S, u), \qquad x - u \in \operatorname{Nor}(S, u) \subset \operatorname{Dual}(S),$$
$$\{u, -u\} \subset \operatorname{Tan}(S, u), \qquad (x - u) \bullet u = 0,$$
$$|x|^2 = |x - u|^2 + |x - (x - u)|^2 \geqq [\delta_S(x)]^2 + [\delta_{\operatorname{Dual}(S)}(x)]^2,$$

and consequently the equation of the lemma holds.

To prove the converse, suppose S is a closed set such that the equation holds whenever $x \in E_n$. Since the equation also holds with S replaced by $\operatorname{Dual}(S)$, one finds that

$$\delta_S(x) = \delta_{\operatorname{Dual}[\operatorname{Dual}(S)]}(x) \quad \text{whenever} \quad x \in E_n,$$

hence $S = \operatorname{Dual}[\operatorname{Dual}(S)]$.

4.17. LEMMA. *If A is a closed subset of E_n, $0 < t < \infty$, $r > 0$ and*

$$\delta_{\operatorname{Tan}(A,a)}(b - a) \leqq |b - a|^2/(2t)$$

whenever $a, b \in A$ with $|a - b| < 2r$, then $\operatorname{reach}(A) \geqq \inf\{r, t\}$.

Proof. Suppose $\delta_A(x) < \inf\{r, t\}$, $a \in A$, $b \in A$,

$$\delta_A(x) = |x - a| = |x - b|,$$

and assume $a = 0$. Then $|b| \leqq |b - x| + |x| < 2r$,

$$x \in \operatorname{Nor}(A, a),\ \operatorname{Tan}(A, a) \subset \operatorname{Dual}(\{x\}) = \{v: v \bullet x \leqq 0\},$$

$$b \bullet \frac{x}{|x|} \leqq \delta_{\operatorname{Dual}(\{x\})}(b) \leqq \delta_{\operatorname{Tan}(A,a)}(b) \leqq |b|^2/2t,$$

$$0 = |b - x|^2 - |x|^2 = |b|^2 - 2b \bullet x \geqq |b|^2(1 - |x|/t) > 0$$

unless $b=0$.

4.18. Theorem. *If A is a closed subset of E_n and $0<t<\infty$, then the following two conditions are equivalent:*

(1) $\operatorname{reach}(A) \geqq t$.

(2) $\delta_{\operatorname{Tan}(A,a)}(b-a) \leqq |b-a|^2/(2t)$ *whenever* $a, b \in A$.

Accordingly

$$\operatorname{reach}(A)^{-1} = \sup \{2|b-a|^{-2}\delta_{\operatorname{Tan}(A,a)}(b-a): a \in A, b \in A, a \neq b\},$$

where $0^{-1}=\infty$ *and* $\infty^{-1}=0$.

Proof. Applying 4.17 with $r=\infty$ one finds that (2) implies (1).

Now assume (1) and suppose $a=0 \in A$, $b \in A$. If $v \in \operatorname{Nor}(A, a)$, then

$$v \bullet (-b) \geqq -|b|^2|v|/2t$$

according to 4.8 (12) and (7), hence

$$\begin{aligned} |b - v|^2 = |b|^2 + |v|^2 - 2b \bullet v &\geqq |b|^2 + |v|^2 - |b|^2|v|/t \\ &\geqq |b|^2 - |b|^4/(4t^2). \end{aligned}$$

Consequently

$$[\delta_{\operatorname{Nor}(A,a)}(b)]^2 \geqq |b|^2 - |b|^4/(4t^2),$$

and it follows from 4.8 (12) and 4.16 that $[\delta_{\operatorname{Tan}(A,a)}(b)]^2 \leqq |b|^4/(4t^2)$.

4.19. Theorem. *If* $A \subset E_n$, $\operatorname{reach}(A) > t > 0$, $s>0$ *and*

$$f: \{x: \delta_A(x) < s\} \to E_m$$

is a univalent continuously differentiable map such that f, f^{-1}, Df are Lipschitzian with Lipschitz constants M, N, P, then

$$\operatorname{reach}[f(A)] \geqq \inf \{sN^{-1}, (Mt^{-1} + P)^{-1}N^{-2}\}.$$

Proof. Suppose $a \in A$, $b \in A$ and $|f(b) - f(a)| < 2sN^{-1}$.

Applying 4.18 choose $u \in \operatorname{Tan}(A, a)$ so that

$$|b - a - u| \leqq |b - a|^2/(2t).$$

Then $Df(a)(u) \in \operatorname{Tan}[f(A), f(a)]$ and

$$|Df(a)(b - a) - Df(a)(u)| \leqq M|b - a|^2/(2t).$$

Furthermore $|b-a| < 2s$, $\delta_A[\lambda a + (1-\lambda)b] < s$ for $0 \leqq \lambda \leqq 1$, hence Taylor's Theorem implies that

$$|f(b) - f(a) - Df(a)(b - a)| \leqq P|b - a|^2/2.$$

Accordingly

$$\delta_{\operatorname{Tan}[f(A),f(a)]}[f(b) - f(a)] \leqq (Mt^{-1} + P)|b - a|^2/2$$
$$\leqq (Mt^{-1} + P)N^2|f(b) - f(a)|^2/2.$$

Use of 4.17 completes the proof.

4.20. REMARK. It may be shown that under the conditions of 4.15 (4) the set $A^{(k-1)}$ is closed and the set $A - A^{(k-1)}$ is a k dimensional manifold locally definable by equations $f_1(x) = 0, \cdots, f_{n-k}(x) = 0$, where $f_1, \cdots, f_{n-k}$ are real valued continuously differentiable functions with linearly independent Lipschitzian gradients.

A related proposition states that a Lipschitzian map $g: E_m \to E_n$ has a Lipschitzian differential if and only if the subset g of $E_m \times E_n$ has positive reach.

Among those subsets A of E_n for which $\operatorname{reach}(A) > 0$ the k dimensional manifolds may be characterized by the property that $\operatorname{Tan}(A, a)$ is a k dimensional vectorspace for each $a \in A$.

If $t > 0$, then the class of all k dimensional submanifolds A of E_n for which $\operatorname{reach}(A) \geqq t$ is closed relative to the Hausdorff metric; likewise closed is the class of all subsets A of E_n such that $\operatorname{reach}(A) \geqq t$, $\dim(A) \leqq k$ and A is not a k dimensional manifold.

4.21. REMARK. Suppose $m \geqq n$, X is an open subset of E_m, $f: X \to E_n$ is a continuously differentiable map, f and Df are Lipschitzian with Lipschitz constants M and P, and

$$Q = \inf\{Jf(x): x \in X\} > 0.$$

If $A \subset E_n$, $\operatorname{reach}(A) > t > 0$, $r > 0$ and

$$E_m \cap \{x: \delta_{f^{-1}(A)}(x) < r\} \subset X,$$

then

$$\operatorname{reach}[f^{-1}(A)] \geqq \inf\{r, QM^{1-n}(M^2t^{-1} + P)^{-1}\}.$$

5. The curvature measures. In this section several versions of Steiner's formula are derived by a modification of the classical method of [W]; the main innovation is the use of the algebra $\Lambda^{**}(E)$. By means of Steiner's formula the curvature measures corresponding to a set with positive reach are defined, and their basic properties are established. The proofs of the cartesian product formula and of the generalized Gauss-Bonnet Theorem were partly suggested by [H, 6.1.9] and by [A; FE1].

5.1. LEMMA. *If $h: E_n \to E_n$ is Lipschitzian, $V \subset E_n$, $h|V$ is univalent, $(h|V)^{-1}$ is Lipschitzian, $a \in V$, $E_n - V$ has L_n density* 0 *at a, and h is differentiable at a, then $Jh(a) > 0$.*

Proof. Let M be a Lipschitz constant for h. Suppose $a=0$,

$$u \in \text{kernel } Dh(a), \qquad |u| = 1, \qquad 0 < \epsilon < 1,$$

and choose $r>0$ so that $|h(ru)-h(a)|<\epsilon r$ and

$$L_n(\{x: |x| < r + \epsilon r\} - V) < \alpha(n)\epsilon^n r^n.$$

Then there exists a point $v\in V$ for which $|v-ru|<\epsilon r$, hence

$$|h(v) - h(a)| \leqq |h(v) - h(ru)| + |h(ru) - h(a)| \leqq M\epsilon r + \epsilon r,$$

$$\frac{|h(v) - h(a)|}{|v - a|} \leqq \frac{M\epsilon r + \epsilon r}{r - \epsilon r} = \frac{(M + 1)\epsilon}{1 - \epsilon}.$$

In view of the arbitrary nature of ϵ, this conflicts with the assumption that $(h|V)^{-1}$ is Lipschitzian.

5.2. LEMMA. *Suppose:*

(1) *P is a k dimensional Riemannian manifold of class* 1.

(2) *$\theta_1, \cdots, \theta_k$ are continuous differential* 1 *forms of P.*

(3) *$e_1, \cdots, e_k, f_1, \cdots, f_{n-k}, g$ are Lipschitzian maps of P into E_n.*

(4) *For $p\in P$, τ_p is the tangent space of P at p.*

(5) *C is a closed subset of E_n.*

(6) *Q is a bounded Borel subset of P, $g(Q)\subset C$, $g|Q$ is univalent, $(g|Q)^{-1}$ is Lipschitzian.*

(7) *If $p\in Q$, then*

$$(\theta_1|\tau_p), \cdots, (\theta_k|\tau_p) \text{ are orthonormal,}$$

$$e_1(p), \cdots, e_k(p), \qquad f_1(p), \cdots, f_{n-k}(p) \text{ are orthonormal,}$$

$$\operatorname{Nor}[C, g(p)] \subset \left\{ \sum_{j=1}^{n-k} z_j f_j(p) : z \in E_{n-k} \right\},$$

$$S(p) = E_{n-k} \cap \left\{ z: |z| = 1 \quad \text{and} \quad \sum_{j=1}^{n-k} z_j f_j(p) \in \operatorname{Nor}[C, g(p)] \right\}.$$

(8) *If $p\in Q$ and $g, f_1, \cdots, f_{n-k}$ are differentiable at p, then*

$$\bigwedge_{i=1}^{k} [(dg|\tau_p) \bullet e_i(p)] \text{ is a positive multiple of } \bigwedge_{i=1}^{k} (\theta_i|\tau_p),$$

$$G(p) = \sum_{i=1}^{k} [(dg|\tau_p) \bullet e_i(p)] \otimes (\theta_i|\tau_p) \in \Lambda^{1,1}(\tau_p),$$

$$F_j(p) = \sum_{i=1}^{k} [(df_j|\tau_p) \bullet e_i(p)] \otimes (\theta_i|\tau_p) \in \Lambda^{1,1}(\tau_p) \qquad \text{for } j = 1, \cdots, n-k,$$

$$u_m(p) = m!^{-1} \int_{S(p)} \left[\sum_{j=1}^{n-k} z_j F_j(p) \right]^m dH^{n-k-1}z \in \Lambda^{m,m}(\tau_p) \qquad \text{for } m = 0, 1, \cdots, k.$$

(9) $0 \leqq r < \operatorname{reach}[C, g(p)]$ *whenever* $p \in Q$.

Under these conditions the following formula holds:

$$L_n(\{x: \delta_C(x) \leqq r \quad and \quad \xi_C(x) \in g(Q)\})$$
$$= \sum_{m=0}^{k} r^{n-k+m}(n-k+m)^{-1} \int_Q \operatorname{trace}[(k-m)!^{-1}G(p)^{k-m}u_m(p)]dH^k p.$$

Proof. Let $h: P \times E_{n-k} \to E_n$,

$$h(p, z) = g(p) + \sum_{j=1}^{n-k} z_j f_j(p) \qquad \text{for } (p, z) \in P \times E_{n-k},$$

$$V = (P \times E_{n-k}) \cap \left\{(p, z): p \in Q, \ |z| \leqq r, \sum_{j=1}^{n-k} z_j f_j(p) \in \operatorname{Nor}[C, g(p)]\right\},$$

$$W = \{x: \delta_C(x) \leqq r, \xi_C(x) \in g(Q)\},$$

and note that h is Lipschitzian. Using 4.8 (13) with $A = C$ and $K = g(Q)$, one also sees that $h(V) = W$, $h|V$ is univalent and $(h|V)^{-1}$ is Lipschitzian. Further let

$$Y: P \times E_{n-k} \to P, \ Y(p, z) = p \text{ for } (p, z) \in P \times E_{n-k},$$

and let $Z_1, \cdots, Z_{n-k}$ be the real valued functions on $P \times E_{n-k}$ such that

$$Z_j(p, z) = z_j \text{ for } (p, z) \in P \times E_{n-k}, \quad j = 1, \cdots, n-k.$$

Accordingly

$$h = (g \circ Y) + \sum_{j=1}^{n-k} Z_j \cdot (f_j \circ Y).$$

If $(p, z) \in Q \times E_{n-k}$ and T is the tangentspace of $P \times E_{n-k}$ at (p, z), then dY maps T onto τ_p, inducing

$$Y^*: \wedge^*(\tau_p) \to \wedge^*(T),$$

and the linear functions

$$Y^*(\theta_1|\tau_p), \cdots, Y^*(\theta_k|\tau_p), \ dZ_1|T, \cdots, dZ_{n-k}|T$$

form an orthogonal basis of $\wedge^1(T)$. If $g, f_1, \cdots, f_{n-k}$ are differentiable at p, then h is differentiable at (p, z),

$$(dh|T) \bullet e_i(p) = Y^*\left[(dg|\tau_p) \bullet e_i(p) + \sum_{j=1}^{n-k} z_j(df_j|\tau_p) \bullet e_i(p)\right]$$

for $i = 1, \cdots, k$ and

$$(dh|T) \bullet f_s(p) = (dZ_s|T) + Y^*\left[(dg|\tau_p) \bullet f_s(p) + \sum_{j=1}^{n-k} z_j(df_j|\tau_p) \bullet f_s(p)\right]$$

for $s=1, \cdots, n-k$, hence

$$\bigwedge_{i=1}^{k} [(dh \mid T) \bullet e_i(p)] \wedge \bigwedge_{s=1}^{n-k} [(dh \mid T) \bullet f_s(p)]$$

$$= Y^*\left(\bigwedge_{i=1}^{k} \left[(dg \mid \tau_p) \bullet e_i(p) + \sum_{j=1}^{n-k} z_j (df_j \mid \tau_p) \bullet e_i(p) \right] \right) \wedge \bigwedge_{s=1}^{n-k} (dZ_s \mid T)$$

$$= \text{trace}\left(k!^{-1} \left[G(p) + \sum_{j=1}^{k} z_j F_j(p) \right]^k \right) \bigwedge_{i=1}^{k} (\theta_i \mid \tau_p) \wedge \bigwedge_{j=1}^{n-k} (dZ_s \mid T),$$

and therefore

$$Jh(p, z) = \left| \text{trace}\left(k!^{-1} \left[G(p) + \sum_{j=1}^{n-k} z_j F_j(p) \right]^k \right) \right|.$$

Now consider the case in which $(p, z) \in V$ and $(P \times E_{n-k}) - V$ has density 0 at (p, z). It follows that if $0 < t \leqq 1$, then $(p, tz) \in V$ and $(P \times E_{n-k}) - V$ has density 0 at (p, tz). Accordingly Lemma 5.1 implies that

$$0 < Jh(p, tz) = \left| \text{trace}\left(k!^{-1} \left[G(p) + \sum_{j=1}^{n-k} tz_j F_j(p) \right]^k \right) \right|$$

for $0 < t \leqq 1$. Since the quantity inside the absolute value signs depends continuously on t, and is positive for $t=0$ by virtue of (8), this quantity is positive for $0 \leqq t \leqq 1$. One infers that in the formula for $Jh(p, z)$ the absolute value signs may be omitted, for H^n almost all (p, z) in V.

Using standard integral formulae and the binomial theorem one finally computes

$$L_n(W) = \int_V Jh(p, z) dH^n(p, z)$$

$$= \int_Q \int_0^r \int_{tS(p)} Jh(p, z) dH^{n-k-1} z dt dH^k p$$

$$= \int_Q \int_0^r t^{n-k-1} \int_{S(p)} Jh(p, tz) dH^{n-k-1} z dt dH^k p$$

$$= \int_Q \int_0^r t^{n-k-1} \int_{S(p)} \text{trace}\left(k!^{-1} \left[G(p) + \sum_{j=1}^{n-k} tz_j F_j(p) \right]^k \right) dH^{n-k-1} z dt dH^k p$$

$$= \int_Q \text{trace}\left(\int_0^r t^{n-k-1} \int_{S(p)} \sum_{m=0}^{k} t^m (k-m)!^{-1} G(p)^{k-m} \right.$$

$$\left. m!^{-1} \left[\sum_{j=1}^{n-k} z_j F_j(p) \right]^m dH^{n-k-1} z dt \right) dH^k p$$

$$= \sum_{m=0}^{k} r^{n-k+m} (n-k+m)^{-1} \int_Q \text{trace}[(k-m)!^{-1} G(p)^{k-m} u_m(p)] dH^k p.$$

5.3. COROLLARY. *Suppose*

(1) *P is a k dimensional submanifold of class* 1 *of E_n.*

(2) *$f_1, \cdots, f_{n-k}$ are Lipschitzian maps of P into E_n.*

(3) *If $p \in P$, then $f_1(p), \cdots, f_{n-k}(p)$ form an orthonormal base of* $\operatorname{Nor}(P, p)$.

(4) *If $p \in P$, then τ_p is the (intrinsic) tangent space of P at p.*

(5) *If $p \in P$ and f_j is differentiable at p, then $F_j(p) \in \wedge^{1,1}(\tau_p)$ and the bilinear form corresponding to $F_j(p)$ is the second fundamental form of P at p associated with the normal vectorfield f_j [mapping $(u, v) \in \tau_p \times \tau_p$ onto $df_j(u) \bullet dg(v)$, where $g: P \to E_n$ by inclusion].*

(6) *C is a closed subset of E_n, $P \subset C$.*

(7) *If $p \in P$, then*

$$S(p) = E_{n-k} \cap \left\{ z: |z| = 1 \text{ and } \sum_{j=1}^{n-k} z_j f_j(p) \in \operatorname{Nor}(C, p) \right\}.$$

(8) *If $p \in P$ and $f_1, \cdots, f_{n-k}$ are differentiable at p, then*

$$u_m(p) = m!^{-1} \int_{S(p)} \left[\sum_{j=1}^{n-k} z_j F_j(p) \right]^m dH^{n-k-1}z \in \wedge^{m,m}(\tau_p)$$

for $m = 0, 1, \cdots, k$.

(9) *Q is a bounded Borel subset of P.*

(10) $0 \leqq r < \operatorname{reach}(C, p)$ *whenever $p \in Q$.*

Under these conditions the following formula holds:

$$L_n(\{x: \delta_C(x) \leqq r \text{ and } \xi_C(x) \in Q\}) = \sum_{m=0}^{k} r^{n-k+m}(n-k+m)^{-1} \int_Q \operatorname{trace}[u_m(p)] dH^k p.$$

Proof. Since both members of the preceding equation are countably additive with respect to Q, the problem is local and one may assume, in view of (2) and (3), that there exist Lipschitzian maps $e_1, \cdots, e_k$ of P into E_n such that if $p \in P$, then $e_1(p), \cdots, e_k(p)$ form an orthonormal base for $\operatorname{Tan}(P, p)$. Letting $\theta_1, \cdots, \theta_k$ be the 1-forms of P such that

$$\theta_i \mid \tau_p = (dg \mid \tau_p) \bullet e_i(p) \qquad \text{for } p \in P, i = 1, \cdots, k,$$

one readily verifies that Lemma 5.2 is applicable; the factor $(k-m)!^{-1} G(p)^{k-m}$ may now be omitted because the bilinear form corresponding to $G(p)$ is now the first fundamental form of P [mapping $(u, v) \in \tau_p \times \tau_p$ onto $dg(u) \bullet dg(v)$].

5.4. DEFINITION. Suppose $A \subset E_n$ and $0 < \delta_A(p) < \operatorname{reach}(A)$. Then $P = \{x: \delta_A(x) = \delta_A(p)\}$ is an $n-1$ dimensional submanifold of class 1 of E_n, with the Lipschitzian unit normal vectorfield $(\operatorname{grad} \delta_A) \mid P$, according to 4.8 (5) and (3). If τ_p is the tangentspace of P at p and $(\operatorname{grad} \delta_A) \mid P$ is differentiable at p, then

$$\Xi_A(p) \in \wedge^{1,1}(\tau_p)$$

is defined by the following condition: The bilinear form corresponding to $\Xi_A(p)$ is the second fundamental form of P at p associated with $(\text{grad}\,\delta_A)|P$.

5.5. THEOREM. *If* $A \subset E_n$, $0 < s < \text{reach}(A)$ *and*

$$A_s = \{x: \delta_A(x) \leqq s\}, \qquad A_s' = \{x: \delta_A(x) \geqq s\}, \qquad P_s = \{x: \delta_A(x) = s\},$$

then the following three statements hold:

(1) *If* $0 \leqq r < \text{reach}(A) - s$ *and* Q *is a bounded Borel subset of* P_s, *then*

$$L_n(\{x: \delta_{A_s}(x) \leqq r \text{ and } \xi_{A_s}(x) \in Q\}) = \sum_{m=0}^{n-1} r^{m+1}(m+1)!^{-1} \int_Q \text{trace}[\Xi_A(p)^m]\,dH^{n-1}p.$$

(2) *If* $0 \leqq r < s$ *and* Q *is a bounded Borel subset of* P_s, *then*

$$L_n(\{x: \delta_{A_s'}(x) \leqq r \text{ and } \xi_{A_s'}(x) \in Q\}) = \sum_{m=0}^{n-1} r^{m+1}(m+1)!^{-1}(-1)^m \int_Q \text{trace}[\Xi_A(p)^m]\,dH^{n-1}p.$$

(3) *If* $0 \leqq r < s$ *and* K *is a bounded Borel subset of* E_n, *then*

$$L_n(\{x: \delta_A(x) \leqq r \text{ and } \xi_A(x) \in K\}) = L_n[A_s \cap \overset{-1}{\xi_A}(K)] + \sum_{m=0}^{n-1} (r-s)^{m+1}(m+1)!^{-1} \int_{P_s \cap \overset{-1}{\xi_A}(K)} \text{trace}[\Xi_A(p)^m]\,dH^{n-1}p.$$

Proof of (1). Apply 5.3 with

$$P = P_s, \quad k = n-1, \quad f_1 = (\text{grad}\,\delta_A)|P_s, \quad F_1 = \Xi_A|P_s, \quad C = A_s,$$
$$S(p) = \{1\}, \quad u_m(p) = m!^{-1}[\Xi_A(p)]^m.$$

Proof of (2). Apply 5.3 with

$$P = P_s, \quad k = n-1, \quad f_1 = (\text{grad}\,\delta_A)|P_s, \quad F_1 = \Xi_A|P_s, \quad C = A_s',$$
$$S(p) = \{-1\}, \quad u_m(p) = m!^{-1}[-\Xi_A(p)]^m.$$

Proof of (3). Observe that

$$L_n[A_r \cap \overset{-1}{\xi_A}(K)] = L_n[A_s \cap \overset{-1}{\xi_A}(K)] - L_n[(A_s - A_r) \cap \overset{-1}{\xi_A}(K)],$$

use 4.9 to verify that

$$\{x: r < \delta_A(x) \leqq s \text{ and } \xi_A(x) \in K\} = \{x: \delta_{A_s'}(x) < s - r \text{ and } \xi_{A_s'}(x) \in P_s \cap \overset{-1}{\xi_A}(K)\}$$

and apply (2) with r replaced by $s-r$.

5.6. THEOREM. *If* $A \subset E_n$ *and* $\text{reach}(A) > 0$, *then there exist unique Radon measures* $\psi_0, \psi_1, \cdots, \psi_n$ *over* E_n *such that, for* $0 \leqq r < \text{reach}(A)$,

$$L_n(\{x:\delta_A(x) \leqq r \text{ and } \xi_A(x) \in K\}) = \sum_{i=0}^{n} r^{n-i}\alpha(n-i)\psi_i(K)$$

whenever K is a Borel subset of E_n, and consequently

$$\int_{\{x:\delta_A(x)\leqq r\}} (f \circ \xi_A)dL_n = \sum_{i=0}^{n} r^{n-i}\alpha(n-i)\int fd\psi_i$$

whenever f is a bounded real valued Baire function on E_n with bounded support.

Proof. Clearly $\psi_0, \cdots, \psi_n$ are uniquely determined as soon as the above equations hold for $n+1$ numbers r. On the other hand, if $0<s<\text{reach}(A)$, then measures ψ_i suitable for $0 \leqq r < s$ may be defined by letting $\alpha(n-i)\psi_i(K)$ be the coefficient of r^{n-i} in 5.5 (3).

5.7. DEFINITION. If $A \subset E_n$ and $\text{reach}(A)>0$, then the Radon measures $\psi_0, \psi_1, \cdots, \psi_n$ described in Theorem 5.6 are the *curvature measures associated with A*. Clearly the supports of these measures are contained in A.

Whenever $\psi_0(A), \psi_1(A), \cdots, \psi_n(A)$ are meaningful, for instance in case A is compact, these numbers are the *total curvatures of A*.

Hereafter the dependence on A will be made explicit by writing

$$\Phi_i(A, K) \text{ for } \psi_i(K), \qquad \Phi_i(A) = \Phi_i(A, A) \text{ for } \psi_i(A),$$

$$\Phi_i(A, f) = \int fd\Phi_i(A, \cdot) \text{ for } \int fd\psi_i$$

whenever K is a Borel subset of E_n and f is a Baire function on E_n; furthermore $|\Phi_i|(A, K)$ will be the total variation of $\Phi_i(A, \cdot)$ over K, and $|\Phi_i|(A) = |\Phi_i|(A, A)$.

5.8. REMARK. If $A \subset E_n$, $\text{reach}(A)>0$ and K is a bounded Borel subset of E_n, then

$$\Phi_n(A, K) = L_n(A \cap K), \qquad \Phi_i(A, K) = \Phi_i(A, K \cap \text{Bdry } A) \text{ for } i < n.$$

The first equation is evident from 5.6, the second from 5.5 (3).

It is clear that if M is a rigid motion of E_n, then

$$\Phi_i[M(A), f] = \Phi_i(A, f \circ M)$$

for $i=0, \cdots, n$ and every Baire function f.

If the conditions of 5.5 hold, then

$$\Phi_i(A_s, f) = \alpha(n-i)^{-1}(n-i)!^{-1}\int_{P_s} f(p) \text{ trace}[\Xi_A(p)^{n-i-1}]dH^{n-1}p$$

whenever $i=0, \cdots, n-1$ and f is a bounded Baire function on E_n with compact support. Also, in case A is compact,

$$\Phi_i(A_s) = \sum_{j=0}^{i} \binom{n-j}{n-i} s^{i-j} \frac{\alpha(n-j)}{\alpha(n-i)} \Phi_j(A) \qquad \text{for } i = 0, \cdots, n,$$

because if $0 \leqq r < s - \text{reach}(A)$, then

$$\sum_{i=0}^{n} r^{n-i}\alpha(n-i)\Phi_i(A_s) = L_n(\{x: \delta_{A_s}(x) \leqq r\})$$

$$= L_n(\{x: \delta_A(x) \leqq r+s\}) = \sum_{j=0}^{n} (r+s)^{n-j}\alpha(n-j)\Phi_j(A).$$

5.9. THEOREM. *Suppose* $\epsilon > 0$. *If* $A_1, A_2, A_3, \cdots$ *and* B *are closed subsets of* E_n *such that* $\text{reach}(A_k) \geqq \epsilon$ *for* $k = 1, 2, 3, \cdots$ *and*

$$\delta_{A_k}(x) \to \delta_B(x) \text{ uniformly for } x \in C \text{ as } k \to \infty$$

whenever C *is a compact subset of* E_n, *then* $\text{reach}(B) \geqq \epsilon$ *and for* $i = 0, 1, \cdots, n$ *the sequence of Radon measures*

$$\Phi_i(A_1, \cdot),\ \Phi_i(A_2, \cdot),\ \Phi_i(A_3, \cdot), \cdots$$

converges weakly to the Radon measure $\Phi_i(B, \cdot)$.

Proof. Recalling 4.13, suppose f is a continuous real valued function on E_n with compact support S, and $0 < r < \epsilon$, $\eta > 0$.

Let $M = \sup\{|f(x)| : x \in S\}$, $C = \{x: \delta_S(x) \leqq r\}$, choose a number ζ such that $0 < \zeta < \epsilon - r$ and

$$L_n[\overset{-1}{\xi_B}(C) \cap \{x: r - \zeta < \delta_B(x) < r + \zeta\}] < \eta/M,$$

let $D = C \cap \{x: \delta_B(x) \leqq r + \zeta\}$, and choose a positive integer K such that if $k \geqq K$, then

$$|\delta_{A_k}(x) - \delta_B(x)| < \zeta \text{ for } x \in C, \qquad |\xi_{A_k}(x) - \xi_B(x)| < r \text{ for } x \in D.$$

Let $E = \{x: \delta_B(x) \leqq r\}$, $F_k = \{x: \delta_{A_k}(x) \leqq r\}$, and note that $E \cap \xi_B^{-1}(S) \subset C$. If $k \geqq K$, then

$$F_k \cap \overset{-1}{\xi_{A_k}}(S) = C \cap F_k \cap \overset{-1}{\xi_{A_k}}(S) \subset D \cap \overset{-1}{\xi_{A_k}}(S) \subset \overset{-1}{\xi_B}(C),$$

$$C \cap E \cap \overset{-1}{\xi_{A_k}}(S) \subset D \cap \overset{-1}{\xi_{A_k}}(S) \subset \overset{-1}{\xi_B}(C),$$

$$C \cap [(F_k - E) \cup (E - F_k)] \subset \{x: r - \zeta < \delta_B(x) < r + \zeta\},$$

$$L_n(\overset{-1}{\xi_{A_k}}(S) \cap [(F_k - C \cap E) \cup (C \cap E - F_k)]) < \eta/M,$$

$$\left| \int_{F_k} (f \circ \xi_{A_k}) dL_n - \int_{C \cap E} (f \circ \xi_{A_k}) dL_n \right| < \eta.$$

Since $\xi_{A_k}(x) \to \xi_B(x)$ uniformly for $x \in C \cap E$, one finds by first letting $k \to \infty$ and then letting $\eta \to 0+$ that

$$\lim_{k\to\infty} \int_{F_k} (f \circ \xi_{A_k}) dL_n = \int_{C \cap E} (f \circ \xi_B) dL_n = \int_E (f \circ \xi_B) dL_n$$

and consequently

$$\lim_{k\to\infty} \sum_{i=0}^{n} r^{n-i}\alpha(n-i)\Phi_i(A_k,f) = \sum_{i=0}^{n} r^{n-i}\alpha(n-i)\Phi_i(B,f).$$

Inasmuch as this equation holds for $n+1$ values of r, it follows that

$$\lim_{k\to\infty} \Phi_i(A_k,f) = \Phi_i(B,f) \qquad \text{for } i = 0, 1, \cdots, n.$$

5.10. REMARK. One sees from 5.9 and 4.14 that if $\epsilon>0$ and $i=0, \cdots, n$, then the function on

$$\{A: 0 \neq A \subset E_n \text{ and } \operatorname{reach}(A) \geqq \epsilon\}$$

mapping A onto $\Phi_i(A, \cdot)$ is continuous, with respect to the topologies of the Hausdorff metric and of weak convergence. While the function mapping A onto $|\Phi_i|(A, \cdot)$ is not continuous, it is true that if K is a compact subset of E_n, then

$$\sup\{ |\Phi_i|(A): A \subset K \text{ and } \operatorname{reach}(A) \geqq \epsilon\} < \infty$$

because weak convergence of measures implies boundedness of their total variations.

If $A \subset E_n$, reach$(A)>0$ and $A_s=\{x: \delta_A(x)\leqq s\}$ for $s>0$, then $\Phi_i(A_s, \cdot)$ converges weakly to $\Phi_i(A, \cdot)$ as $s\to 0+$. Moreover, if K is a compact subset of E_n and $0<t<\operatorname{reach}(A)$, then

$$\sup\{ |\Phi_i|(A_s, K): 0 \leqq s \leqq t\} < \infty.$$

5.11. LEMMA. *In addition to the hypotheses of* 5.2 *suppose*:

(10) $k=n-2$.

(11) μ and ν *are Lipschitzian maps of P into E_n.*

(12) *If $p\in P$, then $\mu(p)$ and $\nu(p)$ are linearly independent unit vectors and*

$$f_1(p) = \frac{\mu(p)+\nu(p)}{|\mu(p)+\nu(p)|}, \qquad f_2(p) = \frac{\mu(p)-\nu(p)}{|\mu(p)-\nu(p)|}.$$

(13) *If $p\in Q$, then* Nor $[C, g(p)]$ *is the closed convex cone generated by $\mu(p)$ and $\nu(p)$.*

(14) *If $p\in P$ and μ, ν are differentiable at p, then*

$$M(p) = \sum_{i=1}^{n-2} [(d\mu\,|\,\tau_p) \bullet e_i(p)] \otimes (\theta_i\,|\,\tau_p) \in \Lambda^{1,1}(\tau_p),$$

$$N(p) = \sum_{i=1}^{n-2} [(d\nu\,|\,\tau_p) \bullet e_i(p)] \otimes (\theta_i\,|\,r_p) \in \Lambda^{1,1}(\tau_p).$$

(15) *If $a>0$, $b>0$, $m=0, \cdots, n-2$ and $j=0, \cdots, m$, then*

$$\sigma(a,b) = E_2 \cap \{z: |z| = 1 \text{ and } z_1/a \geqq |z_2/b|\},$$

$$\Delta_{m,j}(a,b) = [(m-j)!j!]^{-1}\int_{\sigma(a,b)} \left(\frac{z_1}{a}+\frac{z_2}{b}\right)^{m-j}\left(\frac{z_1}{a}-\frac{z_2}{b}\right)^{j} dH^1 z.$$

Under these conditions the following three statements hold:

For each $p \in Q$ where μ and ν are differentiable, and for $m=0, \cdots, n-2$,

$$u_m(p) = \sum_{j=0}^{m} \Delta_{m,j}[\,|\mu(p) + \nu(p)|\,,\ |\mu(p) - \nu(p)|\,]M(p)^{m-j}N(p)^j.$$

For $i=0, \cdots, n-2$,

$$\Phi_i[C, g(Q)] = \alpha(n-i)^{-1}(n-i)^{-1} \int_Q \operatorname{trace}[i!^{-1}G(p)^i u_{n-2-i}(p)]\,dH^{n-2}p.$$

For $i=n-1$ and n, $\Phi_i[C, g(Q)]=0$.

Proof. First suppose $p \in Q$, $a=|\mu(p)+\nu(p)|$, $b=|\mu(p)-\nu(p)|$. Note that if $z \in E_2$, then

$$z_1 f_1(p) + z_2 f_2(p) = \left(\frac{z_1}{a} + \frac{z_2}{b}\right)\mu(p) + \left(\frac{z_1}{a} - \frac{z_2}{b}\right)\nu(p),$$

and that $z \in S(p)$ if and only if $|z|=1$ and the above coefficients of $\mu(p)$ and $\nu(p)$ are non-negative. Accordingly $S(p)=\sigma(a, b)$. In case μ and ν are differentiable at p, one also finds that

$$F_1(p) = [M(p) + N(p)]/a, \qquad F_2(p) = [M(p) - N(p)]/b$$

and consequently

$$z_1 F_1(p) + z_2 F_2(p) = \left(\frac{z_1}{a} + \frac{z_2}{b}\right)M(p) + \left(\frac{z_1}{a} - \frac{z_2}{b}\right)N(p)$$

whenever $z \in E_2$. Therefore the first statement follows from 5.2 (8) and the binomial theorem.

The second and third statements may be obtained from the conclusion of 5.2 by computing the coefficient of r^{n-i}.

5.12. Corollary. *Suppose:*

(1) *Q is an $n-2$ rectifiable Borel subset of E_n.*

(2) *g, μ, ν are Lipschitzian maps of Q into E_n, g is univalent, g^{-1} is Lipschitzian.*

(3) $g(Q) \subset C \subset E_n$, reach$(C)>0$.

(4) *If $p \in Q$, then $\mu(p)$ and $\nu(p)$ are linearly independent unit vectors and* Nor $[C, g(p)]$ *is the closed convex cone generated by $\mu(p)$ and $\nu(p)$.*

(5) *η is a common Lipschitzian constant for g, μ, ν.*

(6) *For $i=0, \cdots, n-2$*

$$d_i = \alpha(n-i)^{-1}(n-i)^{-1}\binom{n-2}{i} 2^{n-2-i}(n-2)^{n-2}\pi\eta^{n-2}.$$

Under these conditions it is true that

$$|\Phi_i|\,[C, g(Q)] \leqq d_i \int_Q [\,|\mu(p) + \nu(p)|^{-1} + |\mu(p) - \nu(p)|^{-1}]^{n-2-i} dH^{n-2}p$$

for $i=0, \cdots, n-2$, *and* $|\Phi_i|\,[C, g(Q)]=0$ *for* $i=n-1, n$.

Proof. In view of a standard decomposition one need only consider the case in which $H^{n-2}(Q)=0$ and the case in which Q is contained in an $n-2$ dimensional submanifold P of class 1 of E_n.

In the first case $H^n(Q \times E_2)=0$, according to [F6, 4.2], and the Lipschitzian function ψ, such that

$$\psi(p, w) = g(p) + w_1\mu(p) + w_2\nu(p) \text{ for } (p, w) \in Q \times E_2,$$

maps $Q \times E_2$ onto a set whose L_n measure is zero and which contains $\{x: \xi_C(x) \in g(Q)\}$. Therefore $|\Phi_i|\,[C, g(Q)]=0$ for $i=0, 1, \cdots, n$.

In the second case Lemma 5.11 is applicable, with

$$\Delta_{m,j}(a, b) \leqq [(m-j)!j!]^{-1}(a^{-1} + b^{-1})^m\pi,$$

$$|M(p)| \leqq (n-2)\eta, \quad |N(p)| \leqq (n-2)\eta, \quad |G(p)| \leqq (n-2)\eta,$$

$$\begin{aligned} |i!^{-1}G(p)^i u_{n-2-i}(p)| &\leqq \binom{n-2}{i}[(n-2)\eta]^i \sum_{j=0}^{n-2-i} [\,|\mu(p) + \nu(p)|^{-1} \\ &\qquad + |\mu(p) - \nu(p)|^{-1}]^{n-2-i}\pi \binom{n-2-i}{j}[(n-2)\eta]^{n-2-i} \\ &= \binom{n-2}{i}[(n-2)\eta]^{n-2}\pi 2^{n-2-i}[\,|\mu(p) + \nu(p)|^{-1} + |\mu(p) - \nu(p)|^{-1}]^{n-2-i}. \end{aligned}$$

5.13. LEMMA. *If* $A \subset E_m$, reach$(A) > 0$, f *is a bounded Baire function on* E_m *with compact support, and* u *is a bounded Baire function on* $\{t: t \geqq 0\}$ *whose support is contained in* $\{t: t < \text{reach}(A)\}$, *then*

$$\int (f \circ \xi_A)\cdot(u \circ \delta_A)dL_m = \Phi_m(A,f)\cdot u(0) + \sum_{j=0}^{m-1} \Phi_j(A,f)\alpha(m-j)(m-j)\int_0^\infty t^{m-j-1}u(t)\,dt.$$

Proof. Since the class of all functions u for which this equation holds is closed to subtraction, addition, scalar multiplication and bounded convergence, it need be verified only for the special case when u is the characteristic function of $\{t: 0 \leqq t \leqq r\}$, where $0 \leqq r < \text{reach}(A)$; but then it reduces to the definition of the curvature measures.

5.14. THEOREM. *For any closed sets* $A \subset E_m$ *and* $B \subset E_n$ *the following statements hold*:

(1) $\delta_{A\times B}(x, y) = [\delta_A(x)^2 + \delta_B(y)^2]^{1/2}$ *whenever* $(x, y) \in E_m \times E_n$.

(2) $\text{Unp}(A \times B) = \text{Unp}(A) \times \text{Unp}(B)$ *and*

$$\xi_{A\times B}(x, y) = (\xi_A(x), \xi_B(y)) \quad \textit{whenever} \quad (x, y) \in \mathrm{Unp}(A \times B).$$

(3) $\mathrm{reach}[A\times B, \ (a, \ b)] = \inf\{\mathrm{reach}(A, \ a), \ \mathrm{reach}(B, \ b)\}$ *whenever* $(a, b) \in A\times B$.

(4) *If* $\mathrm{reach}(A) > 0$, $\mathrm{reach}(B) > 0$ *and* $k = 0, \cdots, n+m$, *then*

$$\Phi_k(A \times B, \cdot) = \sum_{i+j=k} \Phi_i(A, \cdot) \otimes \Phi_j(B, \cdot).$$

Proof. The first three statements are easily verified.

To prove (4) suppose

$$0 < r < \inf\{\mathrm{reach}(A), \mathrm{reach}(B)\},$$

f and g are continuous functions on E_m and E_n with compact support, and $h(x, y) = f(x)g(y)$ for $(x, y) \in E_m \times E_n$. Using the definition of $\Phi_k(A\times B, h)$, the Fubini Theorem, the definition of $\Phi_j(B, g)$, and Lemma 5.13, one obtains

$$\sum_{k=0}^{m+n} r^{m+n-k}\alpha(m + n - k)\Phi_k(A \times B, h)$$

$$= \int_{\{(x,y):\ \delta_{A\times B}(x,y)\leq r\}} (h \circ \xi_{A\times B})d(L_m \otimes L_n)$$

$$= \int_{\{x:\ \delta_A(x)\leq r\}} (f \circ \xi_A)(x) \int_{\{y:\ \delta_B(y)^2\leq r^2-\delta_A(x)^2\}} (g \circ \xi_B)(y)\, dL_n y dL_m x$$

$$= \sum_{j=0}^{n} \alpha(n - j)\Phi_j(B, g) \int_{\{x:\ \delta_A(x)\leq r\}} (f \circ \xi_A)(x)[r^2 - \delta_A(x)^2]^{(n-j)/2} dL_m x$$

$$= \sum_{j=0}^{n} \alpha(n - j)\Phi_j(B, g)\Bigg(\Phi_m(A, f)r^{n-j} + \sum_{i=0}^{m-1} (m - i)\alpha(m - i)\Phi_i(A, f) \cdot \int_0^r t^{m-i-1}(r^2 - t^2)^{(n-j)/2}dt\Bigg)$$

$$= \sum_{j=0}^{n} r^{n-j}\alpha(n - j)\Phi_m(A, f)\Phi_j(B, g) + \sum_{i=0}^{m-1}\sum_{j=0}^{n} r^{m+n-i-j}$$

$$\cdot \int_0^1 u^{m-i-1}(1 - u^2)^{(n-j)/2}du\,\alpha(n - j)\alpha(m - i)(m - i)\Phi_i(A, f)\Phi_j(B, g).$$

Now in the special case when A and B consist of single points of E_p and E_q, and when f and g are the characteristic functions of A and B, the preceding formula reduces to

$$r^{p+q}\alpha(p + q) = r^{p+q} \int_0^1 u^{p-1}(1 - u^2)^{q/2}du\,\alpha(p)\alpha(q)p.$$

Returning to the general case one may use this equation with $p=m-i$ and $q=n-j$ to conclude that

$$\sum_{k=0}^{m+n} r^{m+n-k}\alpha(m+n-k)\Phi_k(A\times B, h)$$
$$= \sum_{i=0}^{m}\sum_{j=0}^{n} r^{m+n-i-j}\alpha(m+n-i-j)\Phi_i(A,f)\Phi_j(B,g).$$

5.15. REMARK. Applying Theorem 5.14 to the special case in which B consists of a single point b, one finds that

$$\Phi_k(A\times\{b\}, X\times\{b\}) = \Phi_k(A, X) \qquad \text{for } k = 0, \cdots, m,$$
$$\Phi_k(A\times\{b\}, X\times\{b\}) = 0 \qquad \text{for } k = m+1, \cdots, m+n,$$

whenever X is a bounded Borel subset of E_m. Accordingly the curvature measures behave naturally under an isometric injection of one Euclidean space into another.

5.16. THEOREM. *Suppose A and B are nonempty closed subsets of E_n and*

$$s = \inf\{\text{reach}(A), \text{reach}(B), \text{reach}(A\cup B)\}.$$

Then the following statements hold:

(1) *If $x\in E_n$, then $\delta_{A\cup B}(x)=\inf\{\delta_A(x), \delta_B(x)\}$,*

$$\delta_{A\cap B}(x) \geqq \sup\{\delta_A(x), \delta_B(x)\}.$$

(2) *If $x\in\text{Unp}(A\cup B)$ and $\delta_A(x)\leqq\delta_B(x)$, then*

$$x\in\text{Unp}(A) \quad \textit{and} \quad \xi_A(x) = \xi_{A\cup B}(x).$$

If $x\in\text{Unp}(A\cup B)$ and $\delta_B(x)\leqq\delta_A(x)$, then

$$x\in\text{Unp}(B) \quad \textit{and} \quad \xi_B(x) = \xi_{A\cup B}(x).$$

(3) *If $x\in\text{Unp}(A)$ and $\delta_B(x)\leqq\delta_A(x)<\text{reach}(A\cup B)$, then*

$$x\in\text{Unp}(A\cap B) \quad \textit{and} \quad \xi_A(x) = \xi_{A\cap B}(x).$$

If $x\in\text{Unp}(B)$ and $\delta_A(x)\leqq\delta_B(x)<\text{reach}(A\cup B)$, then

$$x\in\text{Unp}(A\cap B) \quad \textit{and} \quad \xi_B(x) = \xi_{A\cap B}(x).$$

(4) *If $\sup\{\delta_A(x), \delta_B(x)\}<s$, then*

$$\delta_{A\cap B}(x) = \sup\{\delta_A(x), \delta_B)\}, \qquad x\in\text{Unp}(A\cap B),$$
$$\{\xi_A(x), \xi_B(x)\} = \{\xi_{A\cup B}(x), \xi_{A\cap B}(x)\}.$$

(5) $\text{reach}(A\cap B)\geqq s$.

(6) *If $s>0$ and $i=0, \cdots, n$, then*

$$\Phi_i(A,\cdot)+\Phi_i(B,\cdot) = \Phi_i(A\cup B,\cdot)+\Phi_i(A\cap B,\cdot)$$

Proof. Note that (1), (2) are obvious, and that (4), (5) are trivial consequences of (2), (3).

In order to prove (3) one must show that if

$$x \in \text{Unp}(A),\ a = \xi_A(x), \qquad \delta_B(x) \leqq \delta_A(x) < \text{reach}(A \cup B),$$

then $a \in B$. Observe that

$$\xi_A[a + t(x - a)] = a \quad \text{and} \quad a + t(x - a) \in \text{Int Unp}(A \cup B) \quad \text{for } 0 \leqq t \leqq 1.$$

The assumption that $a \notin B$ would imply that

$$\delta_A[a + t(x - a)] < \delta_B[a + t(x - a)] \qquad \text{for small } t > 0,$$
$$0 < \tau = \sup\{t: \xi_{A \cup B}[a + t(x - a)] = a\},$$

and it would follow from 4.8 (6) that $\tau > 1$, hence $\xi_{A\cup B}(x) = a$; but then $a \in B$, because $\xi_{A\cup B}(x) = \xi_B(x)$ according to (2).

Next, to verify (6), suppose f is a bounded Baire function on E_n with bounded support, $0 < r < s$ and

$$A_r = \{x: \delta_A(x) \leqq r\}, \qquad B_r = \{x: \delta_B(x) \leqq r\}.$$

Using (1), (4), (2) one obtains

$$A_r \cup B_r = \{x: \delta_{A\cup B}(x) \leqq r\}, \qquad A_r \cap B_r = \{x: \delta_{A\cap B}(x) \leqq r\},$$

$$\sum_{i=0}^{n} r^{n-i}\alpha(n - i)[\Phi_i(A, f) + \Phi_i(B, f)]$$
$$= \int_{A_r} (f \circ \xi_A)dL_n + \int_{B_r} (f \circ \xi_B)dL_n$$
$$= \int_{A_r - B_r} (f \circ \xi_A)dL_n + \int_{B_r - A_r} (f \circ \xi_B)dL_n + \int_{A_r \cap B_r} [(f \circ \xi_A) + (f \circ \xi_B)]dL_n$$
$$= \int_{A_r - B_r} (f \circ \xi_{A\cup B})dL_n + \int_{B_r - A_r} (f \circ \xi_{A\cup B})dL_n + \int_{A_r \cap B_r} [(f \circ \xi_{A\cup B}) + (f \circ \xi_{A\cap B})]dL_n$$
$$= \int_{A_r \cup B_r} (f \circ \xi_{A\cup B})dL_n + \int_{A_r \cap B_r} (f \circ \xi_{A\cap B})dL_n$$
$$= \sum_{i=0}^{n} r^{n-i}\alpha(n - i)[\Phi_i(A \cup B, f) + \Phi_i(A \cap B, f)].$$

5.17. Remark. The additivity property expressed by 5.16 (6) is a sharper version of certain properties studied by Blaschke [BL, §43] and Hadwiger [H, 6.12], who used these properties together with invariance under rigid motions and continuity (compare 5.8 and 5.9) to characterize Minkowski's Quermassintegrale

$$W_i(A) = \alpha(i)\binom{n}{i}^{-1}\Phi_{n-i}(A)$$

for compact convex sets A. It would be very interesting to know whether there exists a similar characterization of the curvature measures $\Phi_i(A, \cdot)$ for all sets A such that $\operatorname{reach}(A) > 0$.

5.18. REMARK. The proof of 5.19 will make use of the following classical proposition:

Suppose V is a bounded subregion of E_n, the boundary of V is the union of finitely many disjoint $n-1$ dimensional submanifolds of class 1 of E_n, f is a real valued continuously differentiable function on a neighborhood of the closure of V, and at each point of the boundary of V the exterior normal of V and the gradient of f have a positive inner product. Then the Euler-Poincaré characteristic of the closure of V equals the degree of the map

$$(\operatorname{grad} f) \mid \operatorname{Clos} V: (\operatorname{Clos} V, \operatorname{Bdry} V) \to (E_n, E_n - \{0\}).$$

Furthermore, if $W_1, \cdots, W_k$ are the components of Bdry V, then the above degree equals the sum of the degrees of the maps

$$(\operatorname{grad} f) \mid W_i: W_i \to E_n - \{0\}$$

corresponding to $i = 1, \cdots, k$.

Replacing f by a nearby function of class 2 and with nondegenerate critical points, one may derive this proposition from the Morse theory (see [M, Chapter VI, Theorem 1.2, p. 145]).

5.19. THEOREM. *If* $A \subset E_n$, $\operatorname{reach}(A) > 0$ *and* A *is compact, then* $\Phi_0(A)$ *equals the Euler-Poincaré characteristic of* A.

Proof. Suppose $0 < s < \operatorname{reach}(A)$. Since $\Phi_0(A_s) = \Phi_0(A)$, according to 5.8, and since A is a deformation retract of A_s, it is sufficient to show that $\Phi_0(A_s)$ equals the Euler-Poincaré characteristic of A_s.

Applying 5.18 with $f = (\delta_A)^2$, and observing that

$$\operatorname{grad} \delta_A^2(x) = 2s \operatorname{grad} \delta_A(x) \qquad \text{for } x \in P_s,$$

one sees that the Euler-Poincaré characteristic of A_s equals the sum of the degrees with which

$$(\operatorname{grad} \delta_A) \mid P_s: P_s \to V = \{v: |v| = 1\}$$

maps the components of P_s into V; furthermore this sum may be computed by integrating the Jacobian of the above map over P_s with respect to H^{n-1}, and dividing by $n\alpha(n)$.

Consider a point $p \in P_s$ where $(\operatorname{grad} \delta_A) \mid P_s$ is differentiable. If one identifies the tangentspace τ_p of P_s at p with the "parallel" tangentspace of V at $\operatorname{grad} \delta_A(p)$, the differential of $(\operatorname{grad} \delta_A) \mid P_s$ at p becomes the endomorphism of τ_p corresponding to the bilinear form $\Xi_A(p)$, and using 2.12 (5) one finds that the Jacobian determinant equals

$$\operatorname{trace}[(n-1)!^{-1}\Xi_A(p)^{n-1}].$$

Accordingly the Euler-Poincaré characteristic of A_s equals

$$[n\alpha(n)]^{-1}\int_{P_s}\operatorname{trace}[(n-1)!^{-1}\Xi_A(p)^{n-1}]dH^kp$$

$$= \alpha(n)^{-1}n!^{-1}\int_{P_s}\operatorname{trace}[\Xi_A(p)^{n-1}]dH^kp = \Phi_0(A_s),$$

by virtue of 5.8, as was to be shown.

5.20. REMARK. Suppose $F_1, \cdots, F_l$ are elements of an associative and commutative finite dimensional algebra over the reals, and let

$$\eta(m) = \int_{E_l\cap\{z:\ |z|=1\}}\left(\sum_{j=1}^{l} z_jF_j\right)^m dH^{l-1}z$$

for $m=0, 1, 2, \cdots$. Clearly $\eta(m)=0$ in case m is odd, and

$$\eta(0) = H^{l-1}(E_l\cap\{z:\ |z| = 1\}) = l\alpha(l).$$

If m is a positive even integer, then

$$\eta(m) = l\alpha(l)\,\frac{1}{l}\,\frac{3}{l+2}\cdots\frac{m-1}{l+m-2}\left(\sum_{j=1}^{l}(F_j)^2\right)^{m/2}.$$

In fact, Green's formula implies that

$$\begin{aligned}\eta(m) &= \sum_{i=1}^{l}F_i\int_{E_l\cap\{z.\ |z|=1\}}\left(\sum_{j=1}^{l}z_jF_j\right)^{m-1}z_i dH^{l-1}z\\ &= \sum_{i=1}^{l}F_i\int_{E_l\cap\{z:\ |z|<1\}}(m-1)\left(\sum_{j=1}^{l}z_jF_j\right)^{m-2}F_i dL_l z\\ &= (m-1)\sum_{i=1}^{l}(F_i)^2\int_0^1\int_{E_l\cap\{z:\ |z|=r}\left(\sum_{j=1}^{l}z_jF_j\right)^{m-2}dH^{l-1}zdr\\ &= (m-1)\sum_{i=1}^{l}(F_i)^2\int_0^1 r^{m-2+l-1}\eta(m-2)dr\\ &= \frac{m-1}{l+m-2}\sum_{i=1}^{l}(F_i)^2\eta(m-2).\end{aligned}$$

5.21. REMARK. Consider the special case of 5.3 where C is a k dimensional submanifold of class 2 of E_n, and P is open relative to C. If $p\in P$, then

$$S(p) = E_{n-k}\cap\{z:\ |z| = 1\},$$

hence $u_m(p)$ may be computed by means of 5.20. Applying the formula

$$\Phi_i(C, Q) = \alpha(n-i)^{-1}(n-i)^{-1}\int_Q \operatorname{trace}[u_{k-i}(p)]dH_k p$$

for $i=0, 1, \cdots, k$, one finds that

$$\Phi_k(C, Q) = H^k(Q), \qquad \Phi_i(C, Q) = 0 \text{ in case } k-i \text{ is odd},$$

and that, if $k-i$ is even and positive, then

$$\Phi_i(C, Q) = \alpha(n-i)^{-1}(n-i)^{-1}\int_Q \operatorname{trace}\Big\{(k-i)!^{-1}\alpha(n-k)(n-k) \cdot\frac{1}{n-k}\,\frac{3}{n-k+2}\cdots\frac{k-i-1}{n-i-2}\Big(\sum_{j=1}^{n-k}[F_j(p)]^2\Big)^{(k-i)/2}\Big\}dH^k p$$

$$= (2^{k-i}\pi^{(k-i)/2}[(k-i)/2]!)^{-1}\int_Q \operatorname{trace}\Big\{\Big(\sum_{j=0}^{n-k}[F_j(p)]^2\Big)^{(k-i)/2}\Big\}dH^k p.$$

Accordingly the curvature measures $\Phi_i(C, \cdot)$ are the indefinite integrals, with respect to H^k, of certain scalars algebraically associated with the tensor

$$\sum_{j=1}^{n-k}[F_j(p)]^2 \in \wedge^{2,2}(\tau_p).$$

Furthermore this tensor may be identified, except for a factor $-1/2$, with the classical covariant Riemannian curvature tensor of C. In fact, define $e_1, \cdots, e_k$ and $\theta_1, \cdots, \theta_k$ as in the proof of 5.3 and let

$$e_{k+j} = f_j \qquad \text{for } j = 1, \cdots, n-k.$$

Using the familiar notation of Élie Cartan (see [CA; C3]) one obtains

$$df_j \bullet e_s = de_{k+j} \bullet e_s = \omega_{k+j,s} \qquad \text{for } j = 1, \cdots, n-k \text{ and } s = 1, \cdots, k,$$

$$F_j = \sum_{s=1}^{k}(df_j \bullet e_s)\otimes\theta_s = \sum_{s=1}^{k}\omega_{k+j,s}\otimes\theta_s \qquad \text{for } j = 1, \cdots, n-k,$$

$$\sum_{j=1}^{n-k}(F_j)^2 = \sum_{j=1}^{n-k}\sum_{s=1}^{k}\sum_{t=1}^{k}(\omega_{k+j,s}\wedge\omega_{k+j,t})\otimes(\theta_s\wedge\theta_t)$$

$$= \sum_{s=1}^{k}\sum_{t=1}^{k}\Big(\sum_{j=1}^{n-k}\omega_{k+j,s}\wedge\omega_{k+j,t}\Big)\otimes(\theta_s\wedge\theta_t)$$

$$= \sum_{s=1}^{k}\sum_{t=1}^{k}(-\Omega_{s,t})\otimes(\theta_s\wedge\theta_t)$$

$$= \sum_{s=1}^{k}\sum_{t=1}^{k}\sum_{u=1}^{k}\sum_{v=1}^{k} -\frac{1}{2}R_{s,t,u,v}(\theta_u\wedge\theta_v)\otimes(\theta_s\wedge\theta_t).$$

Computing the trace of the $(k-i)/2$th power of this tensor one arrives at $2^{(k-i)/2}[(k-i)/2]!$ times the scalar H_{k-i} introduced in [WE]. (Note: Weyl's $R^{s,t}_{u,v}$ is the negative of Cartan's $R_{s,t,u,v}$). In case k is even and $i=0$ the above formula for $\Phi_0(C)$ reduces, in view of 5.19, to the Gauss-Bonnet Theorem of [A; AW; C1; FE1].

5.22. REMARK. Assuming $A \subset E_n$ and reach$(A)>0$, let

$$\Psi_i(A,f) = \lim_{t\to 0+} |\Phi_i| (\{x: \delta_A(x) \leq t\}, f)$$

whenever $i=0, \cdots, n$ and f is a continuous real valued function on E_n with compact support; from 5.5 one sees that for $i<n$ this limit equals

$$\alpha(n-i)^{-1}\int_{P_s}(f\circ\xi_A)(p)\left|\sum_{m=n-1-i}^{n-1}(m+1)!^{-1}\binom{m+1}{n-i}(-s)^{m+1-n+i}\operatorname{trace}[\Xi_A(p)^m]\right|dH^{n-1}p.$$

Evidently $\Psi_i(A,f)\geq|\Phi_i|(A,f)$.

Under the conditions of 5.21 one finds that

$$\Psi_i(C,Q) = 0 \quad \text{in case} \quad i > k,$$

and that if $i\leq k$, then

$$\Psi_i(C,Q) = \alpha(n-i)^{-1}(n-i)^{-1}(k-i)!^{-1} \int_Q\int_{E_{n-k}\cap\{z:\ |z|=1\}}\left|\operatorname{trace}\left(\left[\sum_{j=1}^{n-k} z_jF_j(p)\right]^{k-i}\right)\right|dH^{n-k-1}z\,dH^kp.$$

The total absolute curvature $\Psi_0(C,C)$ has been studied in [C3] and [CL], and previously for $k=1$ in the theory of knots (see [MI; FE2]).

6. **The principal kinematic formula.** Within the following proof of this integralgeometric formula, concerning two subsets A and B of E_n with positive reach, one may distinguish three component arguments: First, structural considerations (6.1, 6.2, 6.3, and Parts 1, 2, 3, 10, 11, 18 of 6.11) designed to establish qualitative properties of the intersections of A with the isometric images of B. Second, a most delicate convergence proof (6.3, 6.5, 6.10, and Parts 3, 4, 5, 6, 7, 8, 16 of 6.11) showing that in computing the kinematic integral one may approximate A and B by

$$A_r = \{x: \delta_A(x) \leq r\} \quad \text{and} \quad B_r = \{x: \delta_B(x)\leq r\}.$$

Third, computations (6.6, 6.7, 6.8, 6.9, and Parts 9, 12, 13, 14, 15, 17, 19) dealing mainly with A_r and B_r. In these arguments the theory of Hausdorff measure and rectifiability combines with the results of §§4 and 5 to furnish the foundation, the integral formula 3.1 reduces the global analytic problem to a local algebraic problem, and the tensor algebra $\wedge^{**}(E)$ solves the local problem.

6.1. LEMMA. *Suppose* $f: E_m \to E_n$ *is a Lipschitzian map*, $S \subset \{x: f(x)=0\}$, k *is an integer, and for each* $a \in S$ *there exists a* k *dimensional plane* P *such that* $a \in P$ *and* $f|P$ *has a univalent differential at* a.

Then S *is countably* $m-k$ *rectifiable.*

Proof. In view of [F4, 4.3] it is sufficient to show that if a and P are as stated above, then there exist positive numbers r and η such that

$$S \cap \{x: |x-a| < r \text{ and } |x-a| > (1+\eta^2)^{1/2}\delta_P(x)\}$$

is vacuous.

Let M be a Lipschitz constant for f, choose positive numbers r and s such that

$$|f(p)| \geqq s|p-a| \quad \text{whenever} \quad p \in P \quad \text{and} \quad |p-a| < r,$$

and take $\eta = M/s$. If $x \in S$, $|x-a| < r$ and $p = \xi_P(x)$, then

$$|x-p| = \delta_P(x), \qquad |x-a|^2 = |x-p|^2 + |p-a|^2,$$
$$s|p-a| \leqq |f(p)| = |f(x)-f(p)| \leqq M|x-p|,$$
$$|p-a|^2 \leqq \eta^2|x-p|^2, \qquad |x-a|^2 \leqq (1+\eta^2)|x-p|^2.$$

6.2. LEMMA. *Suppose*

X and Y are separable Riemannian manifolds of class 1,

$\dim X = p$, $\dim Y = q$,

$f: X \times Y \to E_n$ *is a Lipschitzian map*,

$S \subset \{(x, y): f(x, y) = 0\}$, k *is an integer*,

and for each $(a, b) \in S$ *the map*

$$f_a: Y \to E_n, \qquad f_a(y) = f(a, y) \qquad \text{for } y \in Y,$$

is differentiable at b *and* df_a *maps the tangent space of* Y *at* b *onto a* k *dimensional subspace of* E_n.

Then S *is countably* $p+q-k$ *rectifiable.*

Proof. Using coordinate systems, it is easy to reduce the problem to the special case in which $X = E_p$ and $Y = E_q$.

For each $(a, b) \in S$ there exists a k dimensional subspace V of the tangent space of E_q at b such that df_a is univalent on V, and to V corresponds in obvious fashion a plane $P \subset E_p \times E_q$ such that $(a, b) \in P$ and $f|P$ has a univalent differential at (a, b).

6.3. LEMMA. *If* X *is a separable* p *dimensional Riemannian manifold of class* 1 *and*

$$\mu: X \to E_n, \qquad \nu: X \to E_n \cap \{u: |u| = 1\}$$

are Lipschitzian maps, then

$$(X \times G_n) \cap \{(x, R): \mu(x) + R[\nu(x)] = 0\}$$

is countably $p+(n-1)(n-2)/2$ *rectifiable.*

Proof. The map $f: X\times G_n\to E_n$,

$$f(x, R) = \mu(x) + R[\nu(x)] \qquad \text{for } (x, R) \in X \times G_n$$

is Lipschitzian. Furthermore for each $x\in X$ the map

$$f_x: G_n \to E_n, \qquad f_x(R) = f(x, R) \qquad \text{for } R \in G_n$$

is analytic and df_x maps the tangent space of G_n at any $R\in G_n$ onto an $n-1$ dimensional subspace of E_n; in fact f_x is obtained by translation through the constant vector $\mu(x)$ from a classical fibre map of G_n onto the $n-1$ sphere. Accordingly Lemma 6.2 applies, with $q=n(n-1)/2$ and $k=n-1$.

6.4. LEMMA. *If* $v\in E_n$, $w\in E_n$, $|v|=|w|=1$, $m<n-1$, *then*

$$\int_{G_n} |v + R(w)|^{-m} d\phi_n R < \infty.$$

Proof. Letting $S=E_n\cap\{x: |x|=1\}$ one finds (see [F4, 5.5]) that

$$H^{n-1}(S)\int_{G_n} |v + R(w)|^{-m} d\phi_n R = \int_S |v + x|^{-m} dH^{n-1}x.$$

Furthermore let $C(r)=S\cap\{x: |v+x|\leqq r\}$ for $r>0$, and observe that there exists an $M<\infty$ such that

$$H^{n-1}[C(r)] \leqq Mr^{n-1} \quad \text{whenever} \quad r > 0.$$

Consequently

$$\int_S |v + x|^{-m} dH^{n-1}x = \sum_{i=0}^{\infty} \int_{C(2^{1-i})-C(2^{-i})} |v + x|^{-m} dH^{n-1}x$$
$$\leqq \sum_{i=0}^{\infty} 2^{im} M 2^{(1-i)(n-1)} = M2^{n-1}\sum_{i=0}^{\infty} (2^{m-n+1})^i < \infty.$$

6.5. REMARK. Clearly the integral considered in Lemma 6.4 is independent of v and w; denote it by I_m.

In case m is an integer it follows from the binomial theorem and Hölder's inequality, with exponents m/j and $m/(m-j)$, that

$$\int_{G_n} [|v + R(w)|^{-1} + |v - R(w)|^{-1}]^m d\phi_n R$$
$$\leqq \sum_{j=0}^{m} \binom{m}{j} (I_m)^{j/m} (I_m)^{(m-j)/m} = 2^m I_m.$$

Similarly one sees that if $\Delta_{m,j}$ is defined as in 5.11 (15), then the integral

$$c_{m,j} = \int_{G_n} \Delta_{m,j}[\,|v + R(w)|\,,\ \ |v - R(w)|\,]\cdot|v \wedge R(w)|\,d\phi_n R$$

is finite and independent of v and w.

6.6. LEMMA. *If* $A \subset E_n$, $B \subset E_n$, reach$(A)>0$, reach$(B)>0$, reach$(A \cap B) > 0$ *and* C *is a bounded Borel set contained in the interior of* B, *then*

$$\Phi_i(A \cap B, C) = \Phi_i(A, C) \qquad \text{for } i = 0, 1, \cdots, n.$$

Proof. In case C is a compact subset of $A \cap \text{Int}(B)$, $\delta_{A\cap B}(x) = \delta_A(x)$ and $\xi_{A\cap B}(x) = \xi_A(x)$ whenever x is sufficiently close to C.

6.7. LEMMA. *Suppose*:

(1) V *and* W *are vector subspaces of* E_n.

(2) $k = \dim V + \dim W - n > 0$.

(3) Ω *is the space of all isometries* ω *such that the domain and the range of* ω *are* k *dimensional vector subspaces of* V *and* W *respectively.*

(4) G, H *are the orthogonal groups of* V, W.

(5) μ, ν *are the Haar measures of* G, H *such that* $\mu(G) = \nu(H) = 1$.

(6) U *is a* k *dimensional real vector space.*

(7) $e\colon U \to V$ *and* $f\colon U \to W$ *are isometric embeddings.*

(8) V' *and* W' *are the orthogonal complements of* V *and* W *in* E_n.

(9) P *is the orthogonal projection of* E_n *onto* V'.

(10) η *is a real valued* ϕ_n *summable function such that, for* $R \in G_n$, $\eta(R)$ *depends only on* $P \circ R|W'$.

(11) ζ *is a real valued continuous function on* Ω.

Then

$$\int_{G_n} \eta(R)\cdot\zeta[R^{-1}|V \cap R(W)]d\phi_n R = \int_{G_n} \eta\, d\phi_n \cdot \int_{G\times H} \zeta(h \circ f \circ e^{-1} \circ g^{-1}) d(\mu \otimes \nu)(g, h).$$

Proof. The group $G \times H$ operates transitively on Ω according to the rule

$$(g, h)\cdot\omega = h \circ \omega \circ g^{-1} \qquad \text{for } g \in G,\ h \in H,\ \omega \in \Omega.$$

Since $f \circ e^{-1} \in \Omega$, a Haar measure ψ over Ω, invariant under the operation of $G \times H$ and with $\psi(\Omega) = 1$, is given by the formula

$$\psi(\zeta) = \int_{G\times H} \zeta(h \circ f \circ e^{-1} \circ g^{-1}) d(\mu \otimes \nu)(g, h)$$

for every continuous real valued function ζ on Ω.

With $g \in G$ associate $A(g) \in G_n$ so that $A(g)|V = g$ and $A(g)|V'$ is the identity map of V'.

With $h \in H$ associate $B(h) \in G_n$ so that $B(h)|W = h$ and $B(h)|W'$ is the identity map of W'.

Then $G\times H$ operates on G_n according to the rule

$$(g, h)\cdot R = A(g) \circ R \circ B(h)^{-1} \qquad \text{for } g \in G,\ h \in H,\ R \in G_n,$$

the measure ϕ_n, the function η and the open set

$$M = G_n \cap \{R: \dim[V \cap R(W)] = k\}$$

are invariant under the action of $G\times H$, and the continuous map

$$u: M \to \Omega, \qquad u(R) = R^{-1} \mid [V \cap R(W)] \qquad \text{for } R \in M,$$

commutes with the operations of $G\times H$ on M and Ω. Therefore another Haar measure Ψ over Ω, invariant under the operation of $G\times H$, is given by the formula

$$\Psi(\zeta) = \int_M \eta\cdot(\zeta \circ u)d\phi_n$$

for every continuous real valued function ζ on Ω. Using the uniqueness of Haar measure and the fact that $\phi_n(G_n - M) = 0$ one concludes that

$$\Psi = \Psi(\Omega)\cdot\psi = \int_{G_n} \eta d\phi_n\cdot\psi.$$

6.8. LEMMA. *If*

U, V, W are finite dimensional real vector spaces with inner products,
G, H are the orthogonal groups of V, W,
μ, ν are the Haar measures of G, H such that $\mu(G) = \nu(H) = 1$,
e: $U \to V$ and f: $U \to W$ are isometric embeddings,
$M \in \wedge^{p,p}(V)$, $N \in \wedge^{q,q}(W)$, $p + q \leq \dim U$,

then

$$\int_{G\times H} \operatorname{trace}[(g \circ e)^*(M)\cdot(h \circ f)^*(N)]d(\mu \otimes \nu)(g, h)$$
$$= \binom{p+q}{q}\binom{\dim U}{p+q}\binom{\dim V}{p}^{-1}\binom{\dim W}{q}^{-1} \operatorname{trace}(M)\operatorname{trace}(N).$$

Proof. Denote the above integral by $F(M, N)$ and observe that F is a bilinear function invariant under the endomorphisms of $\wedge^{p,p}(V)\times\wedge^{q,q}(W)$ induced by $G\times H$. Applying 2.13 twice one infers that there exists a real number c such that

$$F(M, N) = c \operatorname{trace}(M)\operatorname{trace}(N) \qquad \text{for } M \in \wedge^{p,p}(V), \qquad N \in \wedge^{q,q}(W).$$

To determine c, choose

$$I_U \in \wedge^{1,1}(U), \qquad I_V \in \wedge^{1,1}(V), \qquad I_W \in \wedge^{1,1}(W)$$

so that the corresponding bilinear forms are the inner products of U, V, W and let

$$M = (I_V)^p, \qquad N = (I_W)^q.$$

Then

$$(g \circ e)^*(M) \cdot (h \circ f)^*(N) = (I_U)^{p+q} \qquad \text{for } (g, h) \in G \times H,$$

and it follows from 2.12 (4), applied with $k=0$ and $M=1$, that

$$\frac{\dim(U)!}{[\dim(U) - (p+q)]!} = c \frac{\dim(V)!}{[\dim(V) - p]!} \frac{\dim(W)!}{[\dim(W) - q]!}.$$

6.9. REMARK. Suppose χ and ψ are bounded Baire functions on E_n and μ is a Radon measure over E_n.

If ψ and either χ or μ have bounded supports, then

$$\begin{aligned}\int_{E_n \times G_n} \int \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\mu d(L_n \otimes \phi_n)(z, R) \\ = \int_{G_n} \int_{E_n} \chi(x) \int_{E_n} \psi[R^{-1}(x - z)] dL_n z d\mu x d\phi_n R \\ = \int_{G_n} \int_{E_n} \chi(x) \int_{E_n} \psi(y) dL_n y d\mu x d\phi_n R = \int \chi d\mu \cdot \int \psi dL_n.\end{aligned}$$

Similarly, if χ and either ψ or μ have bounded supports, then

$$\int_{E_n \times G_n} \int (\chi \circ T_z \circ R) \cdot \psi d\mu d(L_n \otimes \phi_n)(z, R) = \int \chi dL_n \cdot \int \psi d\mu.$$

If χ has bounded support and S is a bounded Borel subset of E_n, one may apply the first formula with ψ replaced by the product of ψ and the characteristic function of S, to obtain

$$\int_{E_n \times G_n} \int_{(T_z \circ R)(S)} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\mu d(L_n \otimes \phi_n)(z, R) = \int \chi d\mu \cdot \int_S \psi dL_n.$$

If χ and μ have bounded support and S is any Borel subset of E_n, one may apply the second formula with χ replaced by the product of χ and the characteristic function of S, to obtain

$$\int_{E_n \times G_n} \int_{(R^{-1} \circ T_{-z})(S)} (\chi \circ T_z \circ R) \cdot \psi d\mu d(L_n \otimes \phi_n)(z, R) = \int_S \chi dL_n \cdot \int \psi d\mu.$$

6.10. LEMMA. *If $A \subset E_n$, $B \subset E_n$, A is closed, B is compact and*

$$C(t) = \{x : \delta_A(x) \leq t \text{ and } \delta_B(x) \leq t\}$$

for $t > 0$, then $\delta_{C(t)}(x) \to \delta_{A \cap B}(x)$ uniformly for $x \in E_n$ as $t \to 0+$.

Proof. If $t>0$, then $A\cap B\subset C(t)$, $\delta_{A\cap B}(x)\geqq\delta_{C(t)}(x)$ for $x\in E_n$.

Suppose $\epsilon>0$, let $D=\{x: \delta_{A\cap B}(x)<\epsilon\}$, and observe that the sets $C(t)-D$ are compact and their intersection is empty. It follows that if t is sufficiently small, then $C(t)\subset D$, hence

$$\delta_{C(t)}(x) \geqq \delta_D(x) \geqq \delta_{A\cap B}(x) - \epsilon \qquad \text{for } x \in E_n.$$

6.11. THEOREM. *Suppose*

$$A \subset E_n,\ \text{reach}\ (A) > 0,\ B \subset E_n,\ \text{reach}\ (B) > 0,\ B \textit{ is compact}$$

and $i=0, 1, \cdots, n$. *Then*:

(1) *For* $L_n\otimes\phi_n$ *almost all* (z, R) *in* $E_n\times G_n$,

$$\text{reach}[A \cap (T_z\circ R)(B)] > 0.$$

(2) *If* χ, ψ *are bounded Baire functions on* E_n *and* χ *has compact support, then*

$$\int_{E_n\times G_n}\Phi_i[A \cap (T_z\circ R)(B), \chi\cdot(\psi\circ R^{-1}\circ T_{-z})]d(L_n\otimes\phi_n)(z, R) = \sum_{k+l=n+i}\gamma(n, k, l)\Phi_k(A, \chi)\Phi_l(B, \psi).$$

(3) *If* K *is a compact subset of* E_n, *then*

$$\int_{E_n\times G_n}|\Phi_i|\,[A \cap (T_z\circ R)(B), K]d(L_n\otimes\phi_n)(z, R) < \infty.$$

(4) *If* A *is compact, then*

$$\int_{E_n\times G_n}\Phi_i[A \cap (T_z\circ R)(B)]d(L_n\otimes\phi_n)(z, R) = \sum_{k+l=n+i}\gamma(n, k, l)\Phi_k(A)\Phi_l(B).$$

Proof. For $r>0$ let

$$A_r = \{x: \delta_A(x) \leqq r\}, \qquad B_r = \{y: \delta_B(y) \leqq r\},$$
$$V_r = \{x: \delta_A(x) = r\}, \qquad W_r = \{y: \delta_B(y) = r\}.$$

Choose s and ρ so that $0<s<\rho<\inf\{\text{reach}(A), \text{reach}(B)\}$.

For $t\geqq 0$ let Z_t be the subset of $V_s\times W_s\times G_n$ consisting of those points (x, y, R) for which

$$|\text{grad}\,\delta_A(x) + R[\text{grad}\,\delta_B(y)]| \leqq t \ \text{ or }\ |\text{grad}\,\delta_A(x) - R[\text{grad}\,\delta_B(y)]| \leqq t.$$

For $r\geqq 0$ and $R\in G_n$ let $\zeta_{r,R}: V_s\times W_s\to E_n$,

$$\zeta_{r,R}(x, y) = \frac{s-r}{s}\xi_A(x) + \frac{r}{s}x - R\left[\frac{s-r}{s}\xi_B(y) + \frac{r}{s}y\right].$$

For $r\geqq 0$ let $\zeta_r: V_s\times W_s\times G_n\to E_n\times G_n$,

$$\zeta_r(x, y, R) = (\zeta_{r,R}(x, y), R).$$

In case $i \leqq n-2$ define d_i as in 5.12 (6) with

$$\eta = \sup \{\rho(\rho - s)^{-1}, [1 + \rho(\rho - s)^{-1}]s^{-1}\}$$

and let

$$u(x, y, R) = d_i(|\operatorname{grad} \delta_A(x) + R[\operatorname{grad} \delta_B(y)]|^{-1} + |\operatorname{grad} \delta_A(x) - R[\operatorname{grad} \delta_B(y)]|^{-1})^{n-2-i}$$

for $(x, y, R) \in V_s \times W_s \times G_n$; in case $i > n-2$ let $u(x, y, R) = 0$.

Throughout the following Parts 1 to 7 let K be any compact subset of E_n and let $U = \{x: \delta_K(x) \leqq 3\rho + \operatorname{diam}(B)\} \cap V_s$.

PART 1. *Z_0 is countably $(n+2)(n-1)/2$ rectifiable, and*

$$(L_n \otimes \phi_n)[\zeta_r(Z_0)] = 0 \qquad \textit{for } r \geqq 0.$$

Proof. Applying 6.3 with $X = V_s \times W_s$, $p = 2(n-1)$,

$$\mu(x, y) = \operatorname{grad} \delta_A(x), \qquad \nu(x, v) = \pm \operatorname{grad} \delta_B(y) \quad \text{for}(x, y) \in V_s \times W_s,$$

one finds that Z_0 is countably k rectifiable, where

$$k = 2(n-1) + (n-1)(n-2)/2 = (n+2)(n-1)/2.$$

Since ζ_r is Lipschitzian, by 4.8 (8), it follows that

$$H^{k+1}[\zeta_r(Z_0)] = 0.$$

Furthermore ϕ_n is proportional to the $n(n-1)/2$ dimensional Hausdorff measure over G_n, hence $L_n \otimes \phi_n$ is proportional to the

$$n + n(n-1)/2 = n + (n-1) + (n-1)(n-2)/2 = k + 1$$

dimensional Hausdorff measure over $E_n \times G_n$.

PART 2. *If $0 < r < \rho$, $(z, R) \in E_n \times G_n$ and*

$$g(x, y) = \frac{s - r}{s} \xi_A(x) + \frac{r}{s} x \qquad \text{for } (x, y) \in \overset{-1}{\zeta}_{r,R}\{z\},$$

then $g(\overset{-1}{\zeta}_{r,R}\{z\}) = V_r \cap (T_z \circ R)(W_r)$, g is univalent and Lipschitzian with Lipschitz constant $\rho/(\rho - s)$, and g^{-1} is Lipschitzian.

If $A_r \cap (T_z \circ R)(B_r)$ meets K, then $\overset{-1}{\zeta}_{r,R}\{z\} \subset U \times W_s$.

Proof. The first statement follows from 4.8 (13) and (8), and the fact that, for $c \in V_r \cap (T_z \circ R)(W_r)$,

$$g^{-1}(c) = \left(\frac{r - s}{r} \xi_A(c) + \frac{s}{r} c, \frac{r - s}{r} \xi_B[R^{-1}(c - z)] + \frac{s}{r} R^{-1}(c - z)\right).$$

To prove the second statement, observe that if

$$p \in K \cap A_r \cap (T_z \circ R)(B_r) \quad \text{and} \quad (x, y) \in \overset{-1}{\zeta}_{r,R}\{z\},$$

then

$$\delta_K(x) \leqq |p - x| \leqq |p - g(x, y)| + |g(x, y) - x|$$
$$\leqq \operatorname{diam}(B_r) + |r - s| \leqq \operatorname{diam}(B) + 3\rho.$$

PART 3. *If* $0<r<\rho$, $t\geqq 0$ *and* $(z, R)\in(E_n\times G_n)-\zeta_r(Z_t)$, *then*

$$\operatorname{reach}[A_r \cap (T_z \circ R)(B_r)] > (\rho - r)t/4,$$

$V_r\cap(T_z \circ R)(W_r)$ *is an* $n-2$ *dimensional submanifold of class* 1 *of* E_n, *and*

$$|\Phi_i|\,[A_r \cap (T_z \circ R)(B_r),\, K \cap V_r \cap (T_z \circ R)(W_r)]$$
$$\leqq \int_{(U\times W_s)\cap\zeta_{r,R}^{-1}\{z\}} u(x, y, R)\,dH^{n-2}(x, y).$$

Proof. Observe that

$$(T_z \circ R)(B_r) = \{x\colon (\delta_B \circ R^{-1} \circ T_{-z})(x) \leqq r\},$$
$$\operatorname{grad}(\delta_B \circ R^{-1} \circ T_{-z}) = R \circ (\operatorname{grad} \delta_B) \circ R^{-1} \circ T_{-z},$$

and that if $c\in V_r\cap(T_z \circ R)(W_r)$, $(x, y)\in\overset{-1}{\zeta}_{r,R}\{z\}$, $c=g(x, y)$, where g is the function defined in Part 2, then

$$\operatorname{grad} \delta_A(c) = \operatorname{grad} \delta_A(x), \qquad \operatorname{grad}(\delta_B \circ R^{-1} \circ T_{-z})(c) = R[\operatorname{grad} \delta_B(y)],$$

hence $|\operatorname{grad} \delta_A(c) \pm \operatorname{grad}(\delta_B \circ R^{-1} \circ T_{-z})(c)| > t$.

Now if $c\in A_r\cap(T_z \circ R)(B_r)$ and

$$v \in \operatorname{Nor}(A_r, c), \qquad w \in \operatorname{Nor}[(T_z \circ R)(B_r), c], \qquad |v| > 0, \qquad |w| > 0,$$

then $c\in V_r\cap(T_z \circ R)(W_r)$ and

$$v = |v| \operatorname{grad} \delta_A(c), \qquad w = |w| \operatorname{grad}(\delta_B \circ R^{-1} \circ T_{-z})(c),$$
$$\frac{|v + w|}{|v| + |w|} > \frac{t}{2}$$

because on the line segment joining two unit vectors the midpoint is closest to the origin. Since

$$\operatorname{reach}(A_r) > \rho - r, \qquad \operatorname{reach}[(T_z \circ R)(B_r)] > \rho - r$$

according to 4.9, it follows from 4.10 that

$$\operatorname{reach}[A_r \cap (T_z \circ R)(B_r)] > (t/2)(\rho - r)/2.$$

It is now also clear that $V_r\cap(T_z \circ R)(W_r)$ is an $n-2$ dimensional compact submanifold of class 1 of E_n. Since g^{-1} is Lipschitzian, $\overset{-1}{\zeta}_{r,R}\{z\}$ is $n-2$ rectifiable, and 5.12 may be applied with

$$Q = \overset{-1}{\zeta}_{r,R}\{z\}, \qquad C = A_r \cap (T_z \circ R)(B_r),$$
$$\mu(x, y) = \operatorname{grad} \delta_A(x), \qquad \nu(x, y) = R[\operatorname{grad} \delta_B(y)] \qquad \text{for } (x, y) \in Q.$$

Finally, reference to the last statement of Part 2 completes the proof of Part 3.

PART 4. *If S is a Borel subset of $E_n \times G_n$ and $r \geqq 0$, then*

$$\int_{(U\times W_s\times G_n)\cap \overset{-1}{\zeta}_r(S)} uJ\zeta_r d(H^{n-1} \otimes H^{n-1} \otimes \phi_n)$$
$$= \int_S \int_{(U\times W_s)\cap \overset{-1}{\zeta}_{r,R}\{z\}} u(x, y, R) dH^{n-2}(x, y) d(L_n \otimes \phi_n)(z, R).$$

Proof. Applying Theorem 3.1 with

$X = V_s \times W_s \times G_n, \qquad Y = E_n \times G_n, \qquad f = \zeta_r,$
$g(x, y, R) = u(x, y, R)$ for $(x, y, R) \in (U \times W_s \times G_n) \cap \overset{-1}{\zeta}_r(S)$,
$g(x, y, R) = 0$ for $(x, y, R) \in (V_s \times W_s \times G_n) - (U \times W_s \times G_n) \cap \overset{-1}{\zeta}_r(S)$;
$m = 2(n-1) + n(n-1)/2, \qquad k = n + n(n-1)/2, \qquad m - k = n - 2,$

one obtains

$$\int_{(U\times W_s\times G_n)\cap \overset{-1}{\zeta}_r(S)} u(x, y, R) J\zeta_r(x, y, R) dH^m(x, y, R)$$
$$= \int_S \int_{(U\times W_s\times G_n)\cap \overset{-1}{\zeta}_r\{(z,R)\}} u(x, y, Q) dH^{n-2}(x, y, Q) dH^k(z, R).$$

Furthermore, if $(z, R) \in E_n \times G_n$, then

$$(U \times W_s \times G_n) \cap \overset{-1}{\zeta}_r\{(z, R)\} = \{(x, y, R) : (x, y) \in (U \times W_s) \cap \overset{-1}{\zeta}_{r,R}\{z\}\}$$

and the function mapping (x, y) onto (x, y, R) is an isometry, hence the inside integral equals

$$\int_{(U\times W_s)\cap \overset{-1}{\zeta}_{r,R}\{z\}} u(x, y, R) dH^{n-2}(x, y).$$

Finally let $q = H^{n(n-1)/2}(G_n)$ and observe that

$H^{n(n-1)/2}$ agrees with $q\phi_n$ over G_n,
H^m agrees with $H^{n-1} \otimes H^{n-1} \otimes q\phi_n$ over $U \times W_s \times G_n$,
H^k agrees with $L_n \otimes q\phi_n$ over $E_n \times G_n$.

PART 5. *If $0 < r < \rho$ and S is a Borel subset of $E_n \times G_n$, then*

$$\int_S^* |\Phi_i| [A_r \cap (T_z \circ R)(B_r), K] d(L_n \otimes \phi_n)(z, R)$$

$$\leq \int_{(U \times W_s \times G_n) \cap \zeta_r^{-1}(S)} uJ\zeta_r d(H^{n-1} \otimes H^{n-1} \otimes \phi_n)$$
$$+ [|\Phi_i|(A_r, K) + |\Phi_i|(B_r)](L_n \otimes \phi_n)(S).$$

Proof. One sees from Part 1, 3 and Lemma 6.6 that, for $L_n \otimes \phi_n$ almost all (z, R) in $E_n \times G_n$,

$$|\Phi_i| [A_r \cap (T_z \circ R)(B_r), K]$$
$$\leq |\Phi_i| [A_r \cap (T_z \circ R)(B_r), K \cap V_r \cap (T_z \circ R)(W_r)]$$
$$+ |\Phi_i| [A_r \cap (T_z \circ R)(B_r), K \cap A_r - (T_z \circ R)(W_r)]$$
$$+ |\Phi_i| [A_r \cap (T_z \circ R)(B_r), (T_z \circ R)(B_r) - V_r]$$
$$\leq \int_{(U \times W_s) \cap \zeta_{r,R}^{-1}\{z\}} u(x, y, R) dH^{n-2}(x, y) + |\Phi_i|(A_r, K) + |\Phi_i|(B_r)$$

and then one uses Part 4 to estimate the upper integral over S.

PART 6.

$$\sup_{0 \leq r \leq \rho} \int_{U \times W_s \times G_n} uJ\zeta_r d(H^{n-1} \otimes H^{n-1} \otimes \phi_n) < \infty.$$

Proof. Since the functions ζ_r corresponding to $0 \leq r \leq \rho$ are equi-Lipschitzian, there exists a number M such that

$$J\zeta_r(x, y, R) \leq M \text{ whenever } 0 \leq r \leq \rho,\ x \in U,\ y \in W_s,\ R \in G_n$$

and ζ_r is differentiable at (x, y, R). Assuming $i \leq n-2$ and applying 6.5 with $m = n-2-i$ one finds that the above integrals do not exceed

$$M \int_U \int_{W_s} \int_{G_n} u(x, y, R) d\phi_n R dH^{n-1} y dH^{n-1} x$$
$$\leq M d_i 2^{n-2-i} I_{n-2-i} H^{n-1}(W_s) H^{n-1}(U) < \infty.$$

PART 7. *For each* $\epsilon > 0$ *there exist* $t > 0$, $h > 0$ *and a compact subset* S *of* $E_n \times G_n$ *such that*

$$(L_n \otimes \phi_n)(S) < \epsilon$$

and such that if $0 < r \leq h$, *then*

$$\zeta_r[Z_t \cap (U \times W_s \times G_n)] \subset S,$$

$$\int_S^* |\Phi_i| [A_r \cap (T_z \circ R)(B_r), K] d(L_n \otimes \phi_n)(z, R) < \epsilon.$$

Proof. Recalling 5.10 choose a number $M \geqq 1$ such that

$$|\Phi_i|(A_r, K) + |\Phi_i|(B_r) \leqq M \qquad \text{for } 0 \leqq r \leqq s.$$

Assured by Parts 1 and 4 that

$$[L_n \otimes \phi_n)[\zeta_0(Z_0)] = 0,$$

$$\int_{(U \times W_s \times G_n) \cap \zeta_0^{-1}[\zeta_0(Z_0)]} uJ\zeta_0 d(H^{n-1} \otimes H^{n-1} \otimes \phi_n) = 0,$$

use Part 6 to secure an open subset P of $E_n \times G_n$ such that

$$\zeta_0(Z_0) \subset P, \qquad (L_n \otimes \phi_n)(P) < \epsilon/(2M),$$

$$\int_{(U \times W_s \times G_n) \cap \zeta_0^{-1}(P)} uJ\zeta_0 d(H^{n-1} \otimes H^{n-1} \otimes \phi_n) < \epsilon/2.$$

Choose a compact subset S of P such that

$$\zeta_0[(U \times W_s \times G_n) \cap Z_0] \subset \text{Interior } S,$$

choose a positive number t such that

$$\zeta_0[(U \times W_s \times G_n) \cap Z_t] \subset \text{Interior } S,$$

and choose a positive number $h \leqq s$ such that if $0 \leqq r \leqq h$, then

$$\zeta_r[(U \times W_s \times G_n) \cap Z_t] \subset \text{Interior } S,$$
$$\zeta_r[(U \times W_s \times G_n) - \zeta_0^{-1}(P)] \subset (E_n \times G_n) - S.$$

Since the functions $J\zeta_r$ converge boundedly to $J\zeta_0$ one may also require that if $0 \leqq r \leqq h$ then

$$\int_{(U \times W_s \times G_n) \cap \zeta_0^{-1}(P)} uJ\zeta_r d(H^{n-1} \otimes H^{n-1} \otimes \phi_n) < \epsilon/2.$$

Accordingly, if $0 < r \leqq h$, then

$$(U \times W_s \times G_n) \cap \zeta_r^{-1}(S) \subset \zeta_0^{-1}(P),$$

and it follows from Part 5 that

$$\int_S^* |\Phi_i| [A_r \cap (T_z \circ R)(B_r), K] d(L_n \otimes \phi_n)(z, R) < \epsilon/2 + M\epsilon/(2M) = \epsilon.$$

PART 8. *For $L_n \otimes \phi_n$ almost all (z, R) in $E_n \times G_n$,*

$$\text{reach}[A \cap (T_z \circ R)(B)] > 0$$

and $\Phi_i[A_r \cap (T_z \circ R)(B_r), \cdot]$ converges weakly to $\Phi_i[A \cap (T_z \circ R)(B), \cdot]$ as $r \to 0+$.

Proof. Since these assertions are obviously true in case $A \cap (T_z \circ R)(B)$ is empty, and since E_n is the union of countably many compact sets, it is sufficient to prove that the assertions hold $L_n \otimes \phi_n$ almost everywhere in

$$M(K) = \{(z, R): A \cap (T_z \circ R)(B) \text{ meets } K\},$$

where K is a compact subset of E_n.

Given $\epsilon > 0$, apply Part 7. For $(z, R) \in M(K) - S$ one sees from Parts 2 and 3 that if $0 < r \leqq h$, then

$$\overset{-1}{\zeta_r}\{(z, R)\} \subset U \times W_s \times G_n, \qquad (z, R) \notin \zeta_r(Z_t),$$
$$\text{reach}[A_r \cap (T_z \circ R)(B_r)] > (\rho - h)t/4,$$

and uses 6.10, 4.13, 5.9 to infer that

$$\text{reach}[A \cap (T_z \circ R)(B)] \geqq (\rho - h)t/4$$

and $\Phi_i[A_r \cap (T_z \circ R)(B_r), \cdot]$ converges weakly to $\Phi_i[A \cap (T_z \circ R)(B), \cdot]$ as $r \to 0+$.

In the remaining parts of the proof of the theorem some further conventions are needed:

For $r > 0$, $R \in G_n$ let

$$\eta_{r,R}: V_r \times W_r \to E_n, \qquad \eta_{r,R}(x, y) = x - R(y).$$

For $r > 0$ let η_r: $V_r \times W_r \times G_n \to E_n \times G_n$,

$$\eta_r(x, y, R) = (\eta_{r,R}(x, y), R).$$

For $0 < r < \rho$ let Γ_r be the subset of $V_r \times W_r$ consisting of all points (x, y) such that either $(\text{grad } \delta_A) | V_r$ is not differentiable at x or $(\text{grad } \delta_B) | W_r$ is not differentiable at y.

For $0 < r < \rho$, $x \in V_r$ let $V_r(x)$ be the intrinsic tangent space of V_r at x.

For $0 < r < \rho$, $y \in W_r$ let $W_r(y)$ be the intrinsic tangent space of W_r at y.

For $0 < r < \rho$, $(z, R) \in (E_n \times G_n) - \zeta_r(Z_0)$, $x \in V_r \cap (T_z \circ R)(W_r)$ let $\tau_r(z, R, x)$ be the intrinsic tangent space of $V_r \cap (T_z \circ R)(W_r)$ at x.

For $0 < r < \rho$, $(z, R) \in (E_n \times G_n) - \zeta_r(Z_0)$ let

$$a_{r,z,R}: V_r \cap (T_z \circ R)(W_r) \to V_r, \qquad a_{r,z,R}(x) = x;$$
$$b_{r,z,R}: V_r \cap (T_z \circ R)(W_r) \to W_r, \qquad b_{r,z,R}(x) = (R^{-1} \circ T_{-z})(x).$$

Observe that if $x \in V_r \cap (T_z \circ R)(W_r)$ and $y = (R^{-1} \circ T_{-z})(x)$, then $da_{r,z,R}$ and $db_{r,z,R}$ map $\tau_r(z, R, x)$ isometrically into $V_r(x)$ and $W_r(y)$, and induce homomorphism

$$\overset{*}{a}_{r,z,R} \quad \text{and} \quad \overset{*}{b}_{r,z,R}$$

mapping $\wedge^{**}[V_r(x)]$ and $\wedge^{**}[W_r(y)]$ into $\wedge^{**}[\tau_r(z, R, x)]$.

Defining $\Delta_{m,j}$ and $c_{m,j}$ as in 5.11 (15) and 6.5 let

$$U_j(x, y, R) = \Delta_{n-2-i,j}(|\text{grad } \delta_A(x) + R[\text{grad } \delta_B(y)]|, \\ |\text{grad } \delta_A(x) - R[\text{grad } \delta_B(y)]|)$$

for $j=0, \cdots, n-2-i$, $\delta_A(x)<\rho$, $\delta_B(y)<\rho$, $R\in G_n$. Also let

$$s_j = \binom{n-2-i}{j}\binom{n-2}{n-2-i}\binom{n-1}{n-2-i-j}^{-1}\binom{n-1}{j}^{-1},$$

$$t_j = \alpha(n-i)^{-1}(n-i)^{-1}c_{n-2-i,j}s_j(n-i-j-1)! \\ \cdot\alpha(n-i-j-1)(j+1)!\alpha(j+1).$$

PART 9. *If* $0<r<\rho$, $R\in G_n$, $(x, y)\in V_r\times W_r$, *then*

$$J\eta_{r,R}(x, y) = 2^{(n-2)/2}|\text{grad } \delta_A(x) \wedge R[\text{grad } \delta_B(y)]|.$$

Proof. Since

$$\dim(\tan[V_r, x] \cap \tan[R(W_r), R(y)]) \geqq n-2,$$

there exist $e_1, \cdots, e_n\in E_n$ such that

$e_1, \cdots, e_{n-2}, e_{n-1}$ is an orthonormal base of $\text{Tan}[V_r, x]$,

$e_1, \cdots, e_{n-2}, e_n$ is an orthonormal base of $\text{Tan}[R(W_r), R(y)]$.

Moreover the intrinsic tangent space of $V_r\times W_r$ at (x, y) has an orthonormal base consisting of $2n-2$ vectors which $d\eta_{r,R}$ maps onto

$$e_1, \cdots, e_{n-2}, e_{n-1}, e_1, \cdots, e_{n-2}, e_n$$

respectively, and therefore

$$J\eta_{r,R}(x, y) = 2^{(n-1)/2}|e_1 \wedge \cdots \wedge e_{n-2} \wedge e_{n-1} \wedge e_n| = 2^{(n-1)/2}|e_{n-1} \wedge e_n|.$$

Now the orthogonal complement of $e_1, \cdots, e_{n-2}$ has the two orthonormal bases

$$\{e_{n-1}, \text{grad } \delta_A(x)\} \quad \text{and} \quad \{e_n, R[\text{grad } \delta_B(y)]\},$$

whence it follows that

$$e_{n-1} \wedge e_n = \pm \text{grad } \delta_A(x) \wedge R[\text{grad } \delta_B(y)].$$

PART 10. If $0<r<\rho$, *then*

$$(H^{n-1} \otimes H^{n-1} \otimes \phi_n)[(\eta_r^{-1} \circ \zeta_r)(Z_0)] = 0.$$

Proof. From Theorem 3.1 and Part 1 one obtains

$$\int_{\eta_r^{-1}[\zeta_r(Z_0)]} J\eta_r dH^{2n-2+n(n-1)/2} = \int_{\zeta_r(Z_0)} H^{n-2}[\eta_r^{-1}\{(z, R)\}]dH^{n+n(n-1)/2}(z, R) = 0.$$

Furthermore $J\eta_r$ vanishes almost nowhere, because Part 9 shows that if $(x, y)\in V_r\times W_r$, then

$$J\eta_r(x, y, R) = J\eta_{r,R}(x, y) \neq 0 \qquad \text{for } \phi_n \text{ almost all } R \text{ in } G_n.$$

PART 11. *If* $0<r<\rho$, *then, for* $L_n \otimes \phi_n$ *almost all* (z, R) *in* $E_n \times G_n$,

$$H^{n-2}(\Gamma_r \cap \overset{-1}{\eta}_{r,R}\{z\}) = 0.$$

Proof. Since $H^{2n-2}(\Gamma_r)=0$, $H^{2n-2+n(n-1)/2}(\Gamma_r \times G_n)=0$, and 3.1 implies that

$$\int_{E_n \times G_n} H^{n-2}[(\Gamma_r \times G_n) \cap \eta_r^{-1}\{(x, R)\}]dH^{n+n(n-1)/2}(z, R) = 0.$$

Moreover, if $(z, R) \in E_n \times G_n$, then

$$(\Gamma_r \times G_n) \cap \overset{-1}{\eta_r}\{(z, R)\} = \{(x, y, R): (x, y) \in \Gamma_r \cap \overset{-1}{\eta}_{r,R}\{z\}\}$$

is isometric with $\Gamma_r \cap \overset{-1}{\eta}_{r,R}\{z\}$.

Throughout the following Parts 12 *to* 17 *let* χ *and* ψ *be bounded continuous functions on* E_n, *and suppose* χ *has compact support.*

PART 12. *Suppose* $0<r<\rho$, $(z, R) \in (E_n \times G_n) - \zeta_r(Z_0)$ *and*

$$H^{n-2}(\Gamma_r \cap \overset{-1}{\eta}_{r,R}\{z\}) = 0.$$

If $i \leqq n-2$, *then*

$$\int_{V_r \cap (T_z \circ R)(W_r)} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\Phi_i[A_r \cap (T_z \circ R)(B_r), \cdot]$$
$$= \alpha(n-i)^{-1}(n-i)^{-1}2^{-(n-2)/2} \sum_{j=0}^{n-2-i} \int_{\eta_{r,R}^{-1}\{z\}} \chi(x)\psi(y)U_j(x, y, R)$$
$$\cdot \operatorname{trace}(\overset{*}{a}_{r,z,R}[\Xi_A(x)^{n-2-i-j}]\overset{*}{b}_{r,z,R}[\Xi_B(y)^j])dH^{n-2}(x, y).$$

If $i=n-1$ *or* n, *then* $|\Phi_i|[A_r \cap (T_z \circ R)(B_r), V_r \cap (T_z \circ R)(W_r)]=0$.

Proof. Applying the results of 5.2, 5.3, 5.11 with

$$P = V_r \cap (T_z \circ R)(W_r), \qquad C = A_r \cap (T_z \circ R)(B_r),$$
$$\mu = (\operatorname{grad} \delta_A) \mid P, \qquad \nu = [\operatorname{grad}(\delta_B \circ R^{-1} \circ T_{-z})] \mid P,$$

one finds that

$$\Phi_i(C, Q) = \alpha(n-i)^{-1}(n-i)^{-1} \int_Q \operatorname{trace}[u_{n-2-i}(p)]dH^{n-2}p$$

for every Borel set $Q \subset P$, and consequently

$$\int_P \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\Phi_i(C, \cdot)$$
$$= \alpha(n-i)^{-1}(n-i)^{-1} \int_P \chi(p)\psi[(R^{-1} \circ T_z)(p)] \operatorname{trace}[u_{n-2-i}(p)]dH^{n-2}p.$$

Letting

$$h: E_n \to E_n \times E_n, \qquad h(p) = (p,(R^{-1} \circ T_{-z})(p)) \qquad \text{for } p \in E_n,$$

one sees that $|h(p)-h(q)| = 2^{1/2}|p-q|$ whenever p, $q \in E_n$, and that $h(P) = \eta_{r,R}^{-1}\{z\}$. Hence the preceding integral over P equals

$$2^{-(n-2)/2}\int_{\eta_{r,R}^{-1}\{z\}} \chi(x)\psi(y)\operatorname{trace}[u_{n-2-i}(x)]dH^{n-2}(x, y)$$

$$= 2^{-(n-2)/2}\sum_{j=0}^{n-2-i}\int_{\eta_{r,R}^{-1}\{z\}} \chi(x)\psi(y)U_j(x, y, R)\operatorname{trace}[M(x)^{n-i-2-j}N(x)^j]dH^{n-2}(x, y).$$

Now observe that for $p \in P$ the bilinear forms of $\tau_r(z, R, p)$ corresponding to $M(p)$ and $N(p)$ are the second fundamental forms of P at p associated with the normal vector fields μ and ν; that for $(x, y) \in (V_r \times W_r) - \Gamma_r$ the bilinear forms of $V_r(x)$ and $W_r(y)$ corresponding to $\Xi_A(x)$ and $\Xi_B(y)$ are the second fundamental forms of V_r and W_r at x and y associated with the normal vector fields (grad δ_A)$|V_r$ and (grad δ_B)$|W_r$; and that

$$\mu = [(\operatorname{grad} \delta_A) \mid V_r] \circ a_{r,z,R}, \qquad \nu = R \circ [(\operatorname{grad} \delta_B) \mid W_r] \circ b_{r,z,R}.$$

Since second fundamental forms behave naturally under inclusion maps and isometries, one infers that

$$M(x) = \overset{*}{a}_{r,z,R}[\Xi_A(x)], \qquad N(y) = \overset{*}{b}_{r,z,R}[\Xi_B(y)]$$

whenever $(x, y) \in \eta_{r,R}^{-1}\{z\} - \Gamma_r$.

PART 13. *If* $0 < r < \rho$ *and* $j = 0, \cdots, n-2-i$, *then, for* $H^{n-1} \otimes H^{n-1}$ *almost all* (x, y) *in* $V_r \times W_r$,

$$\int_{G_n} U_j(x, y, R) \mid \operatorname{grad} \delta_A(x) \wedge R[\operatorname{grad} \delta_B(y)] \mid$$
$$\cdot \operatorname{trace}(\overset{*}{a}_{r,x-R(y),R}[\Xi_A(x)^{n-2-i-j}]\overset{*}{b}_{r,x-R(y),R}[\Xi_B(y)^j])d\phi_n R$$
$$= c_{n-2-i,j}s_j \operatorname{trace}[\Xi_A(x)^{n-2-i-j}]\operatorname{trace}[\Xi_B(y)^j].$$

Proof. Using Part 10 one sees that, for $H^{n-1} \otimes H^{n-1}$ almost all (x, y) in $V_r \times W_r$,

$$\phi_n[\{R: (x - R(y), R) \in \zeta_r(Z_0)\}] = 0 \quad \text{and} \quad (x, y) \notin \Gamma_r.$$

Fix such a point (x, y) and let

$$v = \operatorname{grad} \delta_A(x), \qquad w = \operatorname{grad} \delta_B(y), \qquad V = V_r(x), \qquad W = W_r(y),$$
$$M = \Xi_A(x)^{n-2-i-j} \in \Lambda^{n-2-i-j,n-2-i-j}(V), \qquad N = \Xi_B(y)^j \in \Lambda^{j,j}(W),$$
$$\eta(R) = U_j(x, y, R) \; |v \wedge R(w)|$$
$$= \Delta_{n-2-i,j}[\,|v + R(w)|, |v - R(w)|\,] \cdot |v \wedge R(w)| \qquad \text{for } R \in G_n.$$

Recalling 4.6 identify V with $\mathrm{Tan}(V_r, x)$, and W with $\mathrm{Tan}(W_r, y)$.

For ϕ_n almost all R in G_n it is true that

$$(x - R(y), R) \notin \zeta_r(Z_0),$$

and one may identify $\tau_r(x - R(y), R, x)$ with $V \cap R(W)$. Then the restrictions of

$$da_{r,x-R(y),R}, \qquad db_{r,x-R(y),R}$$

to $\tau_r(x - R(y), R, x)$ become identified with

$$\text{the inclusion map of } V \cap R(W) \text{ into } V,$$
$$R^{-1} \mid V \cap R(W): V \cap R(W) \to W.$$

Now readopt the conventions of 6.7 with $k = n - 2$, noting that $\eta(R)$ is determined by $v \bullet R(w)$, hence by $P \circ R \mid W'$. For $\omega \in \Omega$ let $S(\omega)$ be the inclusion map of the domain of ω into V, and let

$$\zeta(\omega) = \mathrm{trace}[S(\omega)^*(M) \cdot \omega^*(N)].$$

Then the given integral can be computed by 6.7, with

$$\int_{G_n} \eta d\phi_n = c_{n-2-i,j}$$

according to 6.5. Furthermore, if $(g, h) \in G \times H$, then

$$\begin{aligned} &dmn(h \circ f \circ e^{-1} \circ g^{-1}) = rng(g \circ e), \\ &\zeta(h \circ f \circ e^{-1} \circ g^{-1}) \\ &\quad = \mathrm{trace}((g \circ e)^*[S(h \circ f \circ e^{-1} \circ g^{-1})^*(M) \cdot (h \circ f \circ e^{-1} \circ g^{-1})^*(N)]) \\ &\quad = \mathrm{trace}[(g \circ e)^*(M) \cdot (h \circ f)^*(N)]. \end{aligned}$$

Accordingly 6.8 implies that

$$\int_{G \times H} \zeta(h \circ f \circ e^{-1} \circ g^{-1}) d(\mu \otimes \nu)(g, h) = s_j \mathrm{trace}(M) \mathrm{trace}(N).$$

PART 14. *If* $0 < r < \rho$ *and* $j = 0, \cdots, n - 2 - i$, *then*

$$\begin{aligned} \int_{E_n \times G_n} \int_{\eta_{r,R}^{-1}\{z\}} &\chi(x)\psi(y) U_j(x, y, R) \\ &\cdot \mathrm{trace}(\overset{*}{a}_{r,z,R}[\Xi_A(x)^{n-2-i-j}] \overset{*}{b}_{r,z,R}[\Xi_A(y)^i]) dH^{n-2}(x, y) d(L_n \otimes \phi_n)(z, R) \\ &= 2^{(n-2)/2} c_{n-2-i,j} s_j (n - i - j - 1)! \alpha(n - i - j - 1)(j + 1)! \alpha(j + 1) \\ &\qquad \cdot \Phi_{i+j+1}(A_r, \chi) \Phi_{n-j-1}(B_r, \psi). \end{aligned}$$

Proof. To see that the above integral exists, apply Theorem 3.1 with $f = \eta_r$, observing that if $(z, R) \in E_n \times G_n$, then $\eta_r^{-1}\{(z, R)\}$ is the isometric image of $\eta_{r,R}^{-1}\{z\}$ under the map carrying (x, y) into (x, y, R).

To compute the integral, first apply Fubini's Theorem to $L_n \otimes \phi_n$, and for each $R \in G_n$ apply 3.1 with $f = \eta_{r,R}$ to obtain

$$\int_{G_n} \int_{V_r \times W_r} \chi(x)\psi(y) U_j(x, y, R) \operatorname{trace}(a^*_{r,x-R(y),R}[\Xi_A(x)^{n-2-i-j}] \\ b^*_{r,x-R(y),R}[\Xi_B(y)^j]) J\eta_{r,R}(x, y) dH^{n-2}(x, y) d\phi_n R.$$

Next apply Fubini's Theorem to $H^{n-2} \otimes \phi_n$, and apply Part 9 to obtain

$$2^{(n-2)/2} \int_{V_r \times W_r} \chi(x)\psi(y) \int_{G_n} U_j(x, y, R) \mid \operatorname{grad} \delta_A(x) \wedge R[\operatorname{grad} \delta_B(y)] \mid \\ \cdot \operatorname{trace}(a^*_{r,x-R(x),R}[\Xi_A(x)^{n-2-i-j}] b^*_{r,x-R(y),R}[\Xi_B(y)^j]) d\phi_n R dH^{2n-2}(x, y).$$

Then apply Part 13, and apply Fubini's Theorem to $H^{n-1} \otimes H^{n-1}$ to obtain

$$2^{(n-2)/2} c_{n-2-i,j} s_j \int_{V_r} \chi(x) \operatorname{trace}[\Xi_A(x)^{n-2-i-j}] dH^{n-1}x \cdot \int_{W_r} \psi(y) \operatorname{trace}[\Xi_B(y)^j] dH^{n-1}y.$$

Finally apply the last formula in 5.8 twice to determine the integrals over V_r and W_r.

PART 15. *If* $0 < r < \rho$ *and* $i \leqq n-1$, *then*

$$\int_{E_n \times G_n} \Phi_i[A_r \cap (T_z \circ R)(B_r), \chi \cdot (\psi \circ R^{-1} \circ T_{-z})] d(L_n \otimes \phi_n)(z, R) \\ = \Phi_i(A_r, \chi)\Phi_n(B_r, \psi) + \Phi_n(A_r, \chi)\Phi_i(B_r, \psi) \\ + \sum_{j=0}^{n-2-i} t_j \Phi_{i+j+1}(A_r, \chi)\Phi_{n-j-1}(B_r, \psi).$$

Proof. If $(z, R) \in (E_n \times G_n) - \zeta_r(Z_0)$, then

$$\Phi_i[A_r \cap (T_z \circ R)(B_r), \chi \cdot (\psi \circ R^{-1} \circ T_{-z})]$$

equals the sum of the three integrals

$$C(z, R) = \int_{V_r \cap (T_z \circ R)(B_r - W_r)} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\Phi_i[A_r \cap (T_z \circ R)(B_r), \cdot],$$

$$D(z, R) = \int_{(A_r - V_r) \cap (T_z \circ R)(W_r)} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\Phi_i[A_r \cap (T_z \circ R)(B_r), \cdot],$$

$$E(z, R) = \int_{V_r \cap (T_z \circ R)(W_r)} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\Phi_i[A_r \cap (T_z \circ R)(B_r), \cdot],$$

and one sees from 6.6 and 5.8 that

$$C(z, r) = \int_{(T_z \circ R)(B_r - W_r)} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\Phi_i(A_r, \cdot),$$

$$D(z, R) = \int_{A_r - V_r} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\Phi_i[(T_z \circ R)(B_r), \cdot]$$

$$= \int_{(R^{-1} \circ T_{-z})(A_r - V_r)} (\chi \circ T_z \circ R) \cdot \psi d\Phi_i(B_r, \cdot).$$

Applying 6.9 one obtains

$$\int_{E_n \times G_n} C d(L_n \otimes \phi_n) = \Phi_i(A_r, \chi)\Phi_n(B_r, \psi),$$

$$\int_{E_n \times G_n} D d(L_n \otimes \phi_n) = \Phi_n(A_r, \chi)\Phi_i(B_r, \psi).$$

If $i \leqq n-2$, it follows from Parts 1, 11, 12, 14 that

$$\int_{E_n \times G_n} E d(L_n \otimes \phi_n) = \sum_{j=0}^{n-2-i} t_j \Phi_{i+j+1}(A_r, \chi)\Phi_{n-j-1}(B_r, \psi).$$

If $i = n-1$, then $E(z, R) = 0$ for $(z, R) \in E_n \times G_n - \zeta_r(Z_0)$.

PART 16. *If* $i \leqq n-1$, *then*

$$\int_{E_n \times G_n} \Phi_i[A \cap (T_z \circ R)(B), \chi \cdot (\psi \circ R^{-1} \circ T_{-z})] d(L_n \otimes \phi_n)(z, R)$$
$$= \Phi_i(A, \chi)\Phi_n(B, \psi) + \Phi_n(A, \chi)\Phi_i(B, \psi) + \sum_{j=0}^{n-2-i} t_j \Phi_{i+j+1}(A, \chi)\Phi_{n-j-1}(B, \psi).$$

Proof. Since one knows from 5.10 that, for $k = 0, \cdots, n$,

$$\Phi_k(A_r, \chi) \to \Phi_k(A, \chi) \quad \text{and} \quad \Phi_k(B_r, \psi) \to \Phi_k(B, \psi)$$

as $r \to 0+$, it will be sufficient to show that the integral of Part 15 approaches the integral of Part 16 as $r \to 0+$.

Let M be a common upper bound of $|\chi|$ and $|\psi|$, and let K be the support of χ.

Given $\epsilon > 0$, choose t, h, S according to Part 7. Then

$$\int_S | \Phi_i[A_r \cap (T_z \circ R)(B_r), \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) | d(L_n \otimes \phi_n)(z, R) < \epsilon M^2$$

for $0 < r \leqq h$, and it follows from Part 8 and Fatou's Lemma that

$$\int_S | \Phi_i[A \cap (T_z \circ R)(B), \chi \cdot (\psi \circ R^{-1} \circ T_{-z})] | d(L_n \otimes \phi_n)(z, R) \leqq \epsilon M^2.$$

Referring again to 5.10 one obtains

$$N = \sup\{|\Phi_i|(C): C \subset B_h \text{ and } \operatorname{reach}(C) \geqq (\rho - h)t/4\} < \infty.$$

If $(z, R) \in (E_n \times G_n) - S$, $0 < r \leqq h$ and $A_r \cap (T_z \circ R)(B_r)$ meets K, then Parts 2 and 3 imply

$$(\rho - h)t/4 < \operatorname{reach}[A_r \cap (T_z \circ R)(B_r)] = \operatorname{reach}[(R^{-1} \circ T_{-z})(A_r) \cap B_r],$$
$$|\Phi_i|[A_r \cap (T_z \circ R)(B_r)] = |\Phi_i|[(R^{-1} \circ T_{-z})(A_r) \cap B_r] \leqq N,$$
$$|\Phi_i[A_r \cap (T_z \circ R)(B_r)], \chi \cdot (\psi \circ R^{-1} \circ T_{-z})| \leqq NM^2.$$

Observing that the set

$$D = \{(z, R): (T_z \circ R)(B_h) \text{ meets } K\} = \{(x - R(y), R): x \in K, R \in G_n, y \in B_h\}$$

is compact, and recalling Part 8, one may apply Lebesgue's theorem concerning bounded convergence to $D - S$, and conclude that

$$\limsup_{r \to 0+} \int_{E_n \times G_n} |\Phi_i[A_r \cap (T_z \circ R)(B_r), \chi \cdot (\psi \circ R^{-1} \circ T_{-z})] - \Phi_i[A \cap (T_z \circ R)(B), \chi \cdot (\psi \circ R^{-1} \circ T_{-z})]| \, d(L_n \otimes \phi_n)(z, R) \leqq 2\epsilon M^2.$$

PART 17.

$$\int_{E_n \times G} \Phi_n[A \cap (T_z \circ R)(B), \chi \cdot (\psi \circ R^{-1} \circ T_{-z})] d(L_n \otimes \phi_n)(z, R) = \Phi_n(A, \chi)\Phi_n(B, \psi).$$

Proof. Using 5.8 and 6.9 one finds that the above integral equals

$$\int_{E_n \times G_n} \int_{A \cap (T_z \circ R)(B)} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) dL_n d(L_n \otimes \phi_n)(z, R)$$
$$= \int_{E_n \times G_n} \int_{(T_z \circ R)(B)} \chi \cdot (\psi \circ R^{-1} \circ T_{-z}) d\Phi_n(A, \cdot) d(L_n \otimes \phi_n)(z, R)$$
$$= \Phi_n(A, \chi) \cdot \int_B \psi dL_n = \Phi_n(A, \chi)\Phi_n(B, \psi).$$

PART 18. PROOF OF (3). In order to prove that the integrand of (3) is an $L_n \otimes \phi_n$ measurable function, consider for $m = 1, 2, 3, \cdots$ the set F_m of all real valued continuous functions f on E_n such that

$$|f(x)| \leqq 1 \text{ whenever } x \in E_n, \qquad f(x) = 0 \text{ whenever } \delta_K(x) \geqq m^{-1},$$

and let C_m be a countable dense (with respect to uniform convergence) subset of F_m. One sees from Parts 16 and 17 that if $f \in F_m$, then

$$\Phi_i[A \cap (T_z \circ R)(B), f]$$

is $L_n \otimes \phi_n$ measurable with respect to (z, R). Furthermore

$$|\Phi_i|[A \cap (T_z \circ R)(B), K] = \lim_{m\to\infty} \sup_{f\in C_m} \Phi_i[A \cap (T_z \circ R)(B), f].$$

Now one may use Part 8, Fatou's Lemma, and Parts 5, 6 to obtain

$$\int_{E_n\times G_n} |\Phi_i|[A \cap (T_z \circ R)(B), K]d(L_n \otimes \phi_n)(z, R)$$

$$\leqq \liminf_{r\to 0+} \int_{E_n\times G_n} |\Phi_i|[A_r \cap (T_z \circ R)(B_r), K]d(L_n \otimes \phi_n)(z, R) < \infty.$$

PART 19. PROOF OF (2) AND (4). Through use of (3) and bounded convergence, the formulae of Parts 16 and 17 may be extended from the continuous case to the case in which χ and ψ are bounded Baire functions, χ having bounded support. Hence the proof of (2), and its corollary (4) resulting when χ and ψ are the characteristic functions of A and B, may be completed by showing that

$$t_j = \gamma(n, i + j + 1, n - j - 1) \qquad \text{for } j = 0, \cdots, n - 2 - i.$$

For this purpose let $k=i+j+1$, $l=n-j-1$ and consider the special case where A and B are k and l dimensional cubes. Using 5.15 one sees that

$$\Phi_k(A) = H^k(A), \qquad \Phi_m(A) = 0 \qquad \text{for } m > k,$$
$$\Phi_l(B) = H^l(B), \qquad \Phi_m(B) = 0 \qquad \text{for } m > l.$$

Moreover, for $L_n \otimes \phi_n$ almost all (z, R) in $E_n \times G_n$, $A \cap (T_z \circ R)(B)$ is either a $k+l-n=i$ dimensional convex set or empty, hence

$$\Phi_i[A \cap (T_z \circ R)(B)] = H^i[A \cap (T_z \circ R)(B)].$$

Substituting in the formula of Part 16 one obtains

$$\int_{E_n\times G_n} H^i[A \cap (T_z \circ R)(B)]d(L_n \otimes \phi_n)(z, R) = t_j H^k(A)H^l(B).$$

On the other hand [F7, 6.2] shows that this integral equals

$$\gamma(n, k, l)H^k(A)H^l(B).$$

6.12. REMARK. The following simple example shows that 6.11 (1) may fail to hold in case neither A nor B is compact.

Let H be the subgroup of E_2 consisting of all points both of whose coordinates are integers, let C be a circle of radius 1/3 in E_2, let A be the union of all translates of C by elements of H, and let B be a straight line in E_2, so that reach$(A)=1/3$ and reach$(B)=\infty$. Then almost all isometric images of B have irrational slopes. Moreover, if L is a straight line with irrational slope, then the image of L in E_2/H is dense in E_2/H, hence L cuts suitable translates of C at arbitrarily near points, and therefore $\text{reach}(A\cap L)=0$.

6.13. THEOREM. *If B is a compact subset of E_n,* $\text{reach}(B)>0$, *ψ is a bounded Baire function on E_n, $i=0, \cdots, n$ and $m=0, \cdots, n-i$, then*

$$\int_{G_n\times E_m} \Phi_i[\lambda_n^{n-m}(R, w) \cap B, \psi]d(\phi_n \otimes L_m)(R, w) = \gamma(n, n-m, m+i)\Phi_{m+i}(B, \psi).$$

Proof. Let

$$A = E_n \cap \{x:x_i = 0 \text{ for } i = 1, \cdots, m\}$$

and let χ be the characteristic function of

$$A \cap \{x: 0 \leqq x_i \leqq 1 \text{ for } i = m+1, \cdots, n\}.$$

Then $\Phi_{n-m}(A,\chi)=1$, $\Phi_j(A,\chi)=0$ for $j\neq n-m$, and the sum of 6.11 (2) equals

$$\gamma(n, n-m, m+i)\Phi_{m+i}(B, \psi).$$

Identifying E_n with $E_m\times E_{n-m}$ and applying the Fubini theorem one finds that the integral of 6.11 (2) equals

$$\int_{E_m\times G_n}\int_{E_{n-m}} \Phi_i[A \cap (T_{(w,y)} \circ R)(B), \chi\cdot(\psi \circ R^{-1} \circ T_{(-w,-y)})$$

$$dL_{n-m}yd(L_m \otimes \phi_n)(w, R).$$

In order to compute inner integral with respect to y, for a fixed (w, R), abbreviate

$$\Phi_i[A \cap (T_{(w,0)} \circ R)(B), \cdot] = \mu, \qquad \psi \circ R^{-1} \circ T_{(-w,0)} = f$$

and note that

$$\Phi_i[A \cap (T_{(w,y)} \circ R)(B), \chi\cdot(\psi \circ R^{-1} \circ T_{(-w,-y)})] = \int (\chi \circ T_{(0,y)})\cdot f d\mu$$

whenever $y\in E_{n-m}$, because $A=T_{(0,y)}(A)$; hence one obtains

$$\int_{E_{n-m}}\int_{E_m\times E_{n-m}} \chi(u, v+y)f(u, v)d\mu(u, v)dL_m y$$

$$= \int_{E_m\times E_{n-m}} f(u, v)\int_{E_{n-m}} \chi(u, v+y)dL_m y d\mu(u, v)$$

$$= \int f d\mu = \Phi_i[(R^{-1} \circ T_{(-w,0)})(A) \cap B, \psi]$$

$$= \Phi_i[\lambda_n^{n-m}(R^{-1}, -w) \cap B, \psi]$$

because if (u, v) belongs to the support of μ, then $(u, v) \in A$, $u=0$, and

$$\int_{E_{n-m}} \chi(0, v + y)\, dL_{n-m}y = 1.$$

Thus one finds that the integral of 6.11 (2) equals

$$\int_{E_m \times G_n} \Phi_i[\overset{n-m}{\lambda_n}(R^{-1}, -w) \cap B, \psi]\, d(L_m \otimes \phi_n)(w, R),$$

and one completes the proof by observing that $L_m \otimes \phi_n$ is invariant under the inversion mapping (w, R) onto $(-w, R^{-1})$.

6.14. REMARK. If $A \subset E_n$, $\operatorname{reach}(A) > 0$ and Q is a bounded Borel subset of $A^{(k)}$ [see 4.15 (3)], then

$$\begin{aligned}
&\Phi_j(A, Q) = 0 \text{ for } j = k+1, \cdots, n,\\
&0 \leqq \Phi_k(A, Q) \leqq H^k(Q),\\
&\Phi_k(A, Q) > 0 \text{ in case } H^k(Q) > 0,\\
&\Phi_k(A, Q) = H^k(Q) \text{ in case } A = A^{(k)}.
\end{aligned}$$

These statements are obviously true for $k=0$, because $A^{(0)}$ is countable and

$$L_n(\{x: \delta_A(x) \leqq r \text{ and } \xi_A(x) = a\}) = r^n \alpha(n) \Phi_0(A, \{a\})$$

whenever $0 < r < \operatorname{reach}(A)$ and $a \in A$. Moreover one may pass from $k=0$ to $k>0$ by means of the following considerations: Recall 6.11 (1), assume A is compact, and let

$$\overset{k}{p_n}: E_n \to E_k, \qquad \overset{k}{p_n}(x) = (x_1, \cdots, x_k) \qquad \text{for } x \in E_n.$$

Using 4.15 (3), verify that

$$Q \cap \overset{n-k}{\lambda_n}(R, w) \subset [A \cap \overset{n-k}{\lambda_n}(R, w)]^{(0)}$$

for $\phi_n \otimes L_k$ almost all (R, w) in $G_n \times E_k$; in fact the set of all those (R, a) in $G_n \times Q$ for which

$$\dim[\operatorname{Nor}(A, a) + R(E_n \cap \{x: x_i = 0 \text{ for } i = k+1, \cdots, n\})] < n$$

has $\phi_n \otimes H^k = H^{n(n-1)/2+k}$ measure 0, and the image of this set under the Lipschitzian map

$$\begin{aligned}
&f: G_n \times Q \to G_n \times E_k,\\
&f(R, a) = (R, (\overset{k}{p_n} \circ R^{-1})(a)) \qquad \text{for } (R, a) \in G_n \times Q,
\end{aligned}$$

contains the set of all those (R, w) for which the above inclusion fails. Now apply 6.13 with B, m, k replaced by A, k, Q; in particular for $i=0$ compare the resulting formula

$$\beta(n, k)\Phi_k(A, Q) = \int_{G_n \times E_k} \Phi_0[A \cap \lambda_n^{n-k}(R, w), Q] d(\phi_n \otimes L_k)(R, w)$$

with the formula

$$\beta(n, k)H^k(Q) = \int_{G_n \times E_k} H^0[Q \cap \lambda_n^{n-k}(R, w)] d(\phi_n \otimes L_k)(R, w)$$

obtained from 4.13 (3) and [F4, 5.14].

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