THE JOHN ELLIPSOID THEOREM

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The following is a lecture given in the functional analysis seminar at the University of South Carolina.

Contents

1. Introduction.

An affine map $\Phi\colon \mathbf{R}^n \to \mathbf{R}^n$ is a map of the form

$$\Phi(x) := Ax + b$$

where $A\colon \mathbf{R}^n \to \mathbf{R}^n$ is linear and $b \in \mathbf{R}^n$ is a constant vector. This is nonsingular iff $\det A \neq 0$. An ***ellipsoid*** in $\mathbf{R}^n$ the image of the closed unit ball B^n of $\mathbf{R}^n$ under a nonsingular affine map. Our goal here is to prove the following famous result of Fritz John.

Theorem 1 (John [3])**.** *Let $K \subset \mathbf{R}^n$ be a convex body (that is a compact convex set with nonempty interior). Then there is an ellipsoid E (called the **John ellipsoid** which will turn out to be the ellipsoid of maximal volume contained in K) so that if c is the center of E then the inclusions*

$$E \subseteq K \subseteq c + n(E - c).$$

hold. (Here $c + n(E - c)$ is the set of points $\{c + n(x - c) : x \in E\}$. This is the dilation of E by a factor of n with center c.)

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Remark 1.1. This result is sharp in the sense that the dilation factor n can not be replaced by a smaller factor. As an example let $x_0, \dots, x_n$ be an affinely independent set of points in $\mathbf{R}^n$ and let K be the convex hull of $\{x_0, \dots, x_n\}$. That is K is an n dimensional simplex in $\mathbf{R}^n$. Then if E is the ellipsoid of maximal volume in K then $E \subseteq K \subseteq c+n(E-c)$, but there is no ellipsoid $E' \subseteq K$ so that $E' \subseteq K \subseteq c'+m(E'-c')$ for any real number $m < n$. For the outline the proof of these claims see Section 4.1. For anther example see Remark 2.3. $\square$

The proof here is one I came up with based loosely on ideas in the expository article [2] of Berger where he gives a similar proof in the case K is symmetric about the origin. For a different proof, which in many ways is preferable to the bare handed approach here, see the article of Keith Ball [1]. John original proof [3] (and I thank Steve Dilworth for giving me a copy of this paper) is quite different and deduces the result from a more general result on maxima and minima of functions subject to inequality constraints very much in spirit of what is now called "geometric programming".

2. Proof of the theorem.

For the rest of this section K will be a convex body in $\mathbf{R}^n$. The basic idea of the proof is to choose $E \subset \mathbf{R}^n$ to be an ellipsoid of maximal volume. Then by an affine change of variables we can assume that E is the unit ball B^n. The proof is completed by showing that if K contains a point p at a distance greater than n from the origin of $\mathbf{R}^n$ then the convex hull of $B^n \cup \{p\}$, and thus also K, contains an ellipsoid of volume greater than B^n which would contradict that B^n has maximal volume. See Figure 1

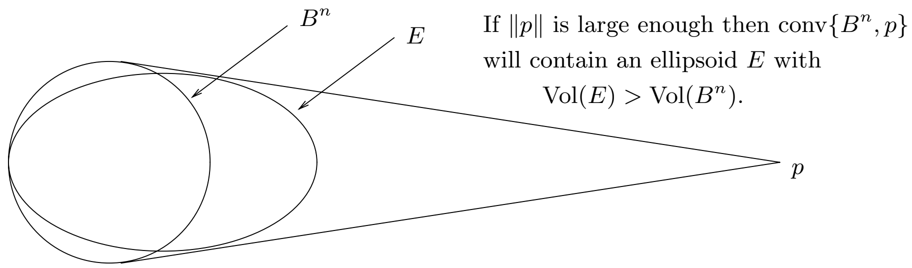

Figure 1

We start by showing that K contains an ellipsoid of maximal volume. If $\Phi(x) = Ax + b$ is an affine max then the volume of the ellipsoid $E = \Phi[B^n]$ is $\operatorname{Vol}(E) = |\det(A)| \operatorname{Vol}(B^n)$. This is a standard piece of affine geometry, but it can also be deduced from the change of variable formula for integrals.

Lemma 2.1. *The convex body K contains an ellipsoid of maximal volume.*

Proof. Let $N := n + n^2$. Then the set of ordered pairs (A, b) where A is an $n \times n$ matrix and $b \in \mathbf{R}^n$ is just $\mathbf{R}^N$. Let $\mathcal{E} := \{(A, b) \in \mathbf{R}^N : AB^n + b \subset$

$K\}$. Then $\mathcal{E}$ is a closed bounded and thus compact subset of $\mathbf{R}^N$ and $(A, b) \mapsto |\det(A)|$ is a continuous function on $\mathcal{E}$. Thus there is an (A_0, b_0) that maximizes this function on $\mathcal{E}$. Then $E := A_0 B^n + b_0$ is the desired ellipsoid. $\square$

If $E = A_0 B^n + b_0$ is an ellipsoid of maximal volume in K then by replacing K by $A_0^{-1}(K - b_0)$ we can assume that $B^n = A_0^{-1}(E - b_0)$ is an ellipsoid of maximal volume in K. Then to prove John's Theorem it is enough to show that if $p \in K$ then $\|p\| \le n$ (where $\|\cdot\|$ is the standard Euclidean norm). The geometric idea of the proof is shown in Figure 1. If $\|p\|$ is large then the convex hull $\operatorname{conv}\{B^n, p\}$ of B^n and p will contain an ellipsoid E with $\operatorname{Vol}(E) > \operatorname{Vol}(B^n)$. But K is convex so $E \subset \operatorname{conv}\{B^n, p\} \subseteq K$ which contradicts that B^n is an ellipsoid of maximal volume in K. What takes some work is showing the critical distance where B^n stops having maximal volume in $\operatorname{conv}\{B^n, p\}$ is $\|p\| > n$.

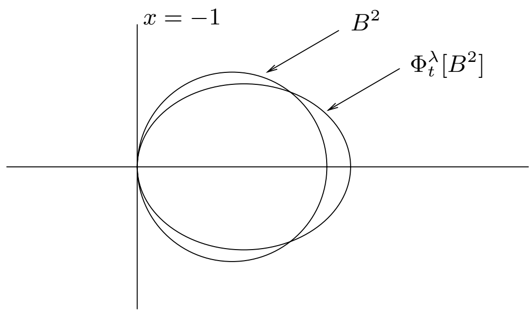

FIGURE 2

We will construct the ellipsoid E as an affine image of the ball B^n. This will first be done in two dimensions and then extended to higher dimensions by symmetry. As it is a truth universally acknowledged that any problem in analysis needs a differential equation to be taken seriously, we will construct our affine maps as flows of solutions to differential equations. Let $\lambda > 0$ then in the plane consider the system of equations

$$\begin{aligned} \dot{x} &= 1 + x \\ \dot{y} &= -\lambda y. \end{aligned}$$

The solution with initial condition $x(0) = x_0$ and $y(0) = y_0$ is

$$\begin{aligned} x(t) &= -1 + e^t(1 + x_0) \\ y(t) &= e^{-\lambda t} y_0. \end{aligned}$$

Therefore if Φ_t^λ is the one parameter group of diffeomorphisms generated by this system of equations (or what is the same thing the one parameter group generated by the vector field $(1 + x)\frac{\partial}{\partial x} - \lambda \frac{\partial}{\partial y}$) is

$$\Phi_t^\lambda(x, y) = (-1 + e^t(1 + x), e^{-\lambda t} y) = (-1 + e^t, 0) + (e^t x, e^{-\lambda t} y)$$

so that for each fixed t the map $(x, y) \mapsto \Phi_t^\lambda(x, y)$ is an affine map. The lines $x = -1$ and $y = 0$ are fixed (set-wise) by Φ_t^λ. The effect of Φ_t^λ on the two dimensional ball B^2 for a small positive value of t is shown in Figure 2.

For $a > 1$ let $C_a := \operatorname{conv}\{B^2, (a, 0)\}$ be the convex hull of the two dimension ball B^2 and the point $(a, 0)$. See Figure 3

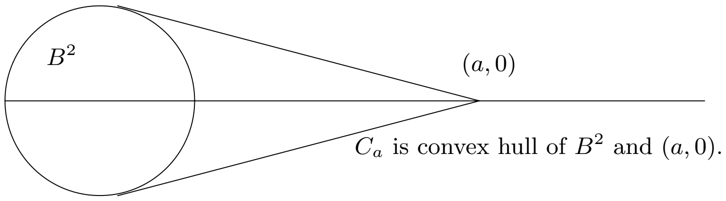

FIGURE 3

The following is the hard step in the proof of John's theorem.

Lemma 2.2. *If* $\lambda > \dfrac{1}{a-1}$ *then* $\Phi_t^\lambda[B^2] \subset C_a$ *for small positive values of* t.

Proof. We first note that the tangent lines to ∂B^2 through $(a, 0)$ (which are part of the boundary of C_a) are $y = \frac{\pm 1}{\sqrt{a^2-1}}(x - a)$. To see this consider the line $y = \frac{-1}{\sqrt{a^2-1}}(x - a)$. Direct calculation shows that both the points $(a, 0)$ and $\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$ are on this line and the point $\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$ is on the unit circle (see Figure 4). The slope of the radius to ∂B^2 ending at $\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$ is $\sqrt{a^2-1}$). The slope of $y = \frac{-1}{\sqrt{a^2-1}}(x - a)$ is $\frac{-1}{\sqrt{a^2-1}}$ which is the negative reciprocal of the slope of the redius to ∂B^2 ending at $\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$. Therefore $y = \frac{-1}{\sqrt{a^2-1}}(x-a)$ is prependicular to the radius where it meets ∂B^2 and this is tangent.

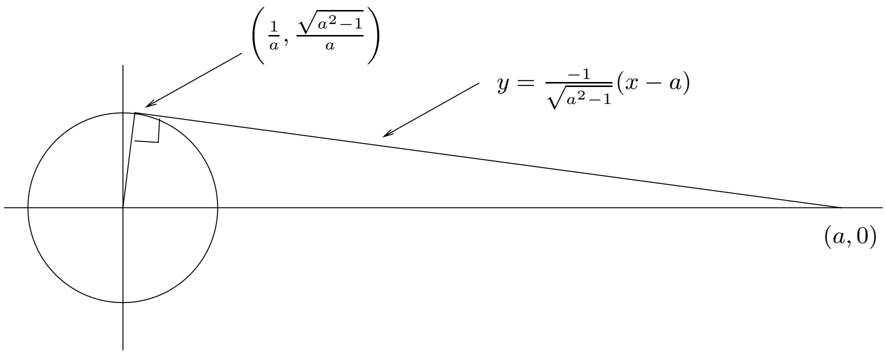

FIGURE 4

Therefore to prove the lemma it is enough to show that for $\lambda > \frac{1}{a-1}$ and small t that $\Phi_t^\lambda\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$ lies below the line $y = \frac{-1}{\sqrt{a^2-1}}(x - a)$. (For as

∂B^2 is tangent to this line at $\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$ it will also be caried into C_a by Φ_t^λ. See Figure 5.)

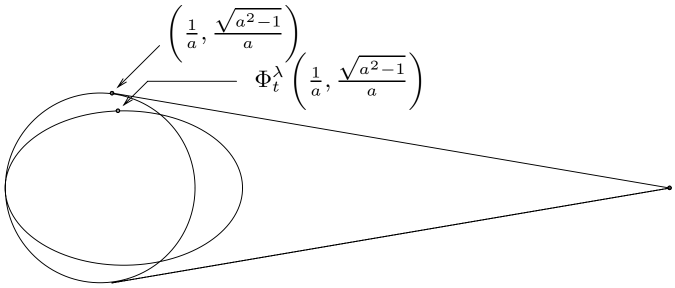

FIGURE 5

Either directly from the definition of Φ_t^λ or from the system of differential equations defining it we have

$$\frac{d}{dt}\Phi_t^\lambda\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)\Bigg|_{t=0} = (\dot{x}(0), \dot{y}(0)) = \left(1+\frac{1}{a}, \frac{-\lambda\sqrt{a^2-1}}{a}\right).$$

Therefore the slope of the tangent to the flow line $t \mapsto \Phi_t^\lambda\big(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\big)$ at $\big(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\big)$ is

$$\frac{\dot{y}(0)}{\dot{x}(0)} = \frac{-\lambda\sqrt{a^2-1}}{a+1}.$$

Therfore $\Phi_t^\lambda\big(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\big)$ will be below the line $y = \frac{-1}{\sqrt{a^2-1}}(x-a)$ provided

$$\frac{-\lambda\sqrt{a^2-1}}{a+1} < \frac{-1}{\sqrt{a^2-1}}.$$

A little algebra shows this is equivalent to $\lambda > \frac{1}{a-1}$. This completes the proof of the lemma □

Proof of Thoerem 1. Assume that $B^n \subset K$ is an ellipsoid of maximal volume in K. Then we need to show that then is not a point p of K with $\|p\| > n$. Assume, toward a contradiction, that there is such a point. Choose orthogonal coordinates $(x, y^1, \ldots, y^{n-1})$ on $\mathbf{R}^n$ so that p is on the positive x axis. Then if $a = \|p\|$ then in these coordinates $p = (a, 0, \ldots, 0)$. The higher dimensional version of C_a used above it $C_a^n := \operatorname{conv}\{B^n, p\} = \operatorname{conv}\{B^n, (a, 0, \ldots, 0)\}$. Define the higher dimensional analogue Φ_t^λ, that is

$$\Psi_t^\lambda(x, y^1, \ldots, y^{n-1}) = (-1 + e^t(x+1), e^{-\lambda t}y^1, \ldots, e^{-\lambda t}y^{n-1}).$$

As before this is an affine map. From Lemma 2.2 and symmetry we have that $\Psi_t^\lambda[B^n] \subset C_a^n \subset K$ for small t provided $\lambda > \frac{1}{a-1}$. The volume of the ellipsoid $\Psi_t^\lambda[B_a^n]$ is

$$\operatorname{Vol}(\Psi_t^\lambda[B^n]) = e^{(1-(n-1)\lambda)t}\operatorname{Vol}(B^n).$$

Now if $a > n$ then it is possible to choose λ with $\lambda > \frac{1}{a-1}$ and $1-(n-1)\lambda > 0$. This implies for small positive t that $\Psi_t^\lambda[B^n] \subset C_a^n \subseteq K$ and $\mathrm{Vol}(\Psi_t^\lambda[B^n]) > \mathrm{Vol}(B^n)$. This contradiction completes the proof of John's theorem. □

Remark 2.3. The proof of the theorem gives anther example showing that the dilation factor n is sharp in John's theorem. If $a = n$ then B^n will be the ellipsoid of maximal volume in C_a and $C_a \subset \alpha B^n$ only for $\alpha \geq n$. □

3. The case of centrally symmetric convex bodies.

When K is symmetric about the origin, that is $-K = K$, then the dilation factor in John's Theorem can be improved.

Theorem 2. *Let K be a convex body in $\mathbf{R}^n$ which is symmetric about the origin. Then there is an ellipsoid E, also symmetric about the origin, so that*

$$E \subseteq K \subseteq \sqrt{n}E.$$

Remark 3.1. The factor $\sqrt{n}$ is this result is sharp. As an example consider $K := \{(x^1, \ldots, x^n) \in \mathbf{R}^n : |x^i| \leq 1\}$ be the unit ball of ℓ_n^∞. Then the unit ball B^n is the ellipsoid of maximal volume in K and the point $P := (1, , \ldots, 1) \in K$ has $\|p\| = \sqrt{n}$ which shows that $\sqrt{n}$ is best possible. □

Remark 3.2. Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be two finite n-dimensional vector spaces. Then define the ***Banach-Mazur distance*** between the two spaces as

$$\mathrm{dist}^{BM}(X,Y) := \inf_{T\colon X \to Y} \|T\|\|T^{-1}\|$$

where T ranges over all linear isomorphisms of $T\colon X \to Y$ and $\|\cdot\|$ is the operator norm. Then a restatement of Theorem 2 is that for and n dimensional X that $\mathrm{dist}\, BM(X, \ell_n^2) \leq \sqrt{n}$ and Remark 3.1 implies that $\mathrm{dist}^{BM}(\ell_n^\infty, \ell_n^2) = \mathrm{dist}(\ell_n^1, \ell_n^2) = \sqrt{n}$ so that this is the sharp bound. □

Proof. The proof is very much like the general case. A variant on Lemma 2.1 shows there as a symmetric ellipsoid $E \subset K$ that has maximal volume. By an affine change of coordinates we can assume that the ball B^n is the ellipsoid of maximal volume in K.

Again we start by looking at a two dimensional problem. This time let $S_a := \mathrm{conv}(B^2 \cup \{(a,0), (-a,0)\}$ (See Figure 6.) As we wish to preserve the symmetry of figures this time we look at the flow of the equations

$$\dot{x} = x$$
$$\dot{y} = -\lambda y$$

where $\lambda > 0$. This has as flow

$$\Phi_t^\lambda(x,y) = (e^t x, e^{-\lambda t} y)$$

which is linear. Again we want to know when $\Phi_t^\lambda[B^2] \subset S_a$ for small t. As in the proof of Lemma 2.2 looking at what happens to the point $\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$

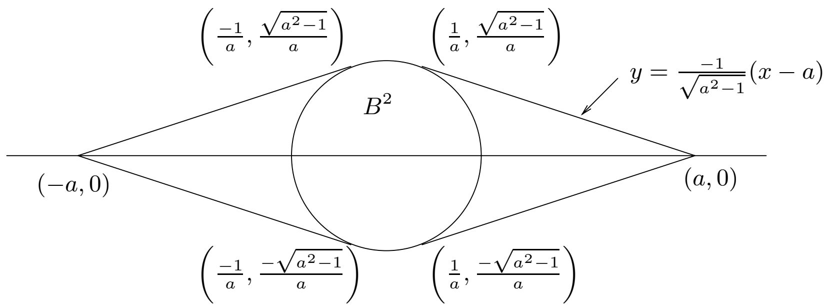

FIGURE 6

is the key. We have

$$\frac{d}{dt}\Phi_t^\lambda\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)\Bigg|_{t=0} = (\dot{x}(0), \dot{y}(0)) = \left(\frac{1}{a}, \frac{-\lambda\sqrt{a^2-1}}{a}\right).$$

Thus the condition for $\Phi_t^\lambda\left(\frac{1}{a}, \frac{\sqrt{a^2-1}}{a}\right)$ to be below the line $y = \frac{-1}{\sqrt{a^2-1}}(x-a)$ for small t is that

$$\frac{\dot{y}(0)}{\dot{y}(0)} = -\lambda\sqrt{a^2-1} < \frac{-1}{\sqrt{a^2-1}}.$$

Some algebra then shows that $\Phi_t^\lambda[B^2] \subset S_a$ for small t if $\lambda ? \frac{1}{a^2-1}$.

To complete the proof if K is a symmetric convex body so that B^n is the ellipsoid of maximum volume contained in K then we wish to show that any $p \in K$ satisfies $\|p\| \leq \sqrt{n}$. We can choose coordinates $x, y^1, \dots, y^{n-1}$ so that p is on the positive x-axis. Then $p = (a, 0, \dots, 0)$. By symmetry $-p = (a, 0, \dots, 0)$ is also in K. Let $S_a^n = \operatorname{conv}(B^n \cup \{p, -p\})$ and define a one parameter group of linear maps acting on $\mathbf{R}^n$ by

$$\Psi_t^\lambda(x, y^1, \dots, y^{n-1}) = (e^t x, e^{-\lambda t} y^1, \dots, e^{-\lambda t} y^{n-1}).$$

Then by symmetry $\Psi_t^\lambda[B^n] \subset S_a \subseteq K$ for small t if $\lambda > \frac{1}{a^2-1}$. The volume of $\Psi_t^\lambda[B^n]$ is

$$\operatorname{Vol}(\Psi_t^\lambda[B^n]) = e^{(1-(n-1)\lambda)t}\operatorname{Vol}(B^n).$$

Now if $a = \|p\| > \sqrt{n}$ then it is possible to choose λ so that $\lambda > \frac{1}{a^2-1}$ (so that $\Psi_t^\lambda[B^n] \subset S_a \subseteq K$ for small t) and $1-(n-1)\lambda > 0$ (so that $\operatorname{Vol}(\Psi_t^\lambda[B^n]) > \operatorname{Vol}(B^n)$. This is a contradiction and completes the proof of Theorem 2. $\square$

4. PROOF OF THE UNIQUNESS OF THE JOHN ELLIPSOID.

Theorem 3. *If $K \subset \mathbf{R}^n$ is a compact body then the ellipsoid of maximal volume is unique.*

Lemma 4.1. *Let E be an ellipsoind in $\mathbf{R}^n$, so that $E = AB^n + b$ with A linear and $b \in \mathbf{R}^n$. Then in this representation we can take A to be positive definite.*

Proof. It is a standard result from linear algebra (the "polar decomposition") that any nonsingular matrix can be expresses as a product $A = PU$ where P is positive definite and U is orthogonal. For any orthogonal matrix we have $UB^n = B^n$. Thus $AB^n = PUB^n = PB^n$. Thus we can replace A by P in the representation $E = AB^n + b$ and assume that A is positive definite. $\square$

Proposition 4.2. *Let A and B be $n \times n$ positive definite matrices. Then*

$$\det(A+B)^{\frac{1}{n}} \geq \det(A)^{\frac{1}{n}} + \det(B)^{\frac{1}{n}}$$

with equality iff there is a positive constant c so that $B = cA$. In particular if equality holds and $\det(A) = \det(B)$ then $A = B$.

Proof. As A is positive definite it has a positive definite square root P. That is $A = P^2$. Then

$$\det(A+B)^{\frac{1}{n}} = \det(P^2+B)^{\frac{1}{n}} = \det(P)^{\frac{2}{n}} \det(I + P^{-1}BP^{-1})^{\frac{1}{n}}$$

and

$$\det(A)^{\frac{1}{n}} + \det(B)^{\frac{1}{n}} = \det(P)^{\frac{2}{n}}(1 + \det(P^{-1}AP^{-1})^{\frac{1}{n}}).$$

Thus letting $C = P^{-1}BP^{-1}$ it is enough to show $\det(I+C)^{\frac{1}{n}} \geq 1+\det(C)^{\frac{1}{n}}$ with equality iff $C = cI$. (If $C = cI$ then $P^{-1}BP^{-1} = cI$ so that $B = cP^2 = cA$.) Now let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of C. Then $\det(I + C)^{\frac{1}{n}} \geq 1 + \det(C)^{\frac{1}{b}}$ is equivalent to

$$\left(\prod_{k=1}^{n}(1+\lambda_k)\right)^{\frac{1}{n}} \geq 1 + (\lambda_1 \cdots \lambda_n)^{\frac{1}{n}} \tag{4.1}$$

with equality iff $\lambda_k = c$ for all k and some positive c. Let $\sigma_k = \sigma_k(\lambda_1, \dots, \lambda_n)$ be the k elementary symmetric function of $\lambda_1, \dots, \lambda_n$. That is

$$\sigma_k(\lambda_1, \dots, \lambda_n) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

Then

$$\prod_{k=1}^{n}(1+\lambda_k) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n.$$

(Note $\sigma_n = \lambda_1 \cdots \lambda_n$.) By the inequality between the arithmetic and geometric means we have

$$\begin{aligned}
\sigma_k(\lambda_1, \dots, \lambda_n) &= \binom{n}{k} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \frac{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}}{\binom{n}{k}} \\
&\ge \binom{n}{k} \prod_{1 \le i_1 < i_2 < \cdots < i_k \le n} (\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k})^{\frac{1}{\binom{n}{k}}} \\
&= \binom{n}{k} (\lambda_1\lambda_2\cdots\lambda_n)^{\frac{\binom{n-1}{k-1}}{\binom{n}{k}}} \\
&= \binom{n}{k} (\lambda_1\lambda_2\cdots\lambda_n)^{\frac{k}{n}}
\end{aligned}$$

and equality will hold iff $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} = \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_k}$ for all subsets $\{i_1, \dots, i_k\}$ and $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$ of size k. This is the case iff there is a c so that $\lambda_i = c$ for all i.

Using this inequality we have

$$\begin{aligned}
(1 + (\lambda_1 \cdots \lambda_n)^{\frac{1}{n}})^n &= \sum_{k=0}^{n} \binom{n}{k} (\lambda_1 \cdots \lambda_n)^{\frac{k}{n}} \\
&\le 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n \\
&= \prod_{k=1}^{n} (1 + \lambda_k)
\end{aligned}$$

with equality iff $\lambda_k = c$ for some $c > 0$. But this inequality is equivalent to the inequality (4.1). This completes the proof. □

Proof of Theorem 3. As above let $\mathcal{P}_n$ be the set of $n \times n$ positive definite matrices and let $\mathcal{K} := \mathcal{P}_n \times \mathbf{R}^n$. Let $\mathcal{E} := \{(A, b) \in \mathcal{K} : AB^n + b \subseteq K\}$. By Lemma 4.1 every ellipsoid contained in K is of the form $AB^n + b$ for some $(A, b) \in \mathcal{K}$. Also as K is convex the set $\mathcal{E}$ is also convex. Let $E_1 = A_1 B^n + b_1$ and $E_2 = A_2 B^n + b_2$ be two ellipsoids of maximal volume contained in K. The volume of the ellipsoid $AB^n + b$ is $\det(A)\operatorname{Vol}(B^n)$, so as E_1 and E_2 are both maximal we have $\det(A_1) = \det(A_2)$. Let $A_3 = \frac{1}{2}(A_1 + A_2)$ and $b_3 = \frac{1}{2}(b_1 + b_2)$. Then as $\mathcal{E}$ is convex the ellipsoid $E_3 = A_3 B^n + b_3 \subseteq K$. But from Proposition 4.2

$$\det(A_3)^{\frac{1}{n}} = \frac{1}{2}\det(A_1 + A_2)^{\frac{1}{n}} \ge \frac{1}{2}(\det(A_1)^{\frac{1}{n}} + \det(A_2)^{\frac{1}{n}}) = \det(A_1)^{\frac{1}{n}}.$$

But as E_1 has maximal volume this implies $\det(A_3) = \det(A_1) = \det(A_2)$ so that equality holds in the inequality. By Proposition 4.2 this implies $A_1 = A_2$. That is E_2 is a translate of E_1.

If only remains to show $b_1 = b_2$. If $b_1 \ne b_2$ then K will contain the convex hull of $E_1 \cup E_2$. Let $b_3 = \frac{1}{2}(b_1 + b_2)$. Then $\operatorname{conv}(E_1 \cup E_2)$ will contain an ellipsoid E_3 centered and b_3 that has volume $\operatorname{Vol}(E_3) > \operatorname{Vol}(E_1)$ as it will

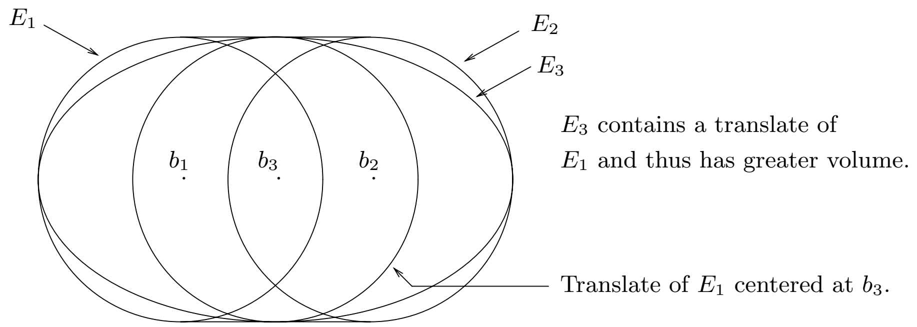

FIGURE 7

contain a translate of E_1 centered at b_3 (see Figure 7 where we have done an affine transformation so that we can assume that E_1 and E_2 are balls.)

But this contradicts that E_1 had maximal volume and so $b_1 = b_2$. This completes the proof. □

4.1. **Examples of where the inequalities are sharp.** Above we we mentioned some examples there the inequalities are sharp. However to show that this is the case by direct calculation is not easy so we show here that the uniqueness result Theorem 3 can be used to find the ellipsoid of maximum volume in case where symmetry is present. Consider the case of a the standard regular simplex σ^n. The easiest realization of this is to view the affine space $\mathbf{R}^n$ as the hyperplane in $\mathbf{R}^{n+1}$ defined by $x^1+x^2+\cdots+x^{n+1} = 1$. Then let $e_1, \dots, e_{n+1}$ be the standard basis of $\mathbf{R}^{n+1}$. Then σ^n is the convex hull of $e_1, \dots, e_{n+1}$. That is σ is the subset of $\mathbf{R}^{n+1}$ defined by $x^1 + \cdots + x^{n+1} = 1$ and $x^i \geq 0$. Let S_{n+1} be the permutation group on $n+1$ objects. Then S_{n+1} acts on $\sigma_n \subset \mathbf{R}^n$ by permuting its vertices $e_1, \dots, e_{n+1}$. This action is by affine maps and in fact by isometries. This implies that this action maps ellipsoids to ellipsoids and that it preserves their volume. As the ellipsoid of maximal volume in σ^n is unique this implies that it is left fixed by all elements of S_{n+1}. However it is not hard to check that the only ellipsoids in $\mathbf{R}^n$ invariant under all elements are the balls centered at $(\frac{1}{n+1}, \dots, \frac{1}{n+1})$. Using this and that all simplexes are affinely equivalent it is not hard to verify the claims in Remark 1.1.

The claims of Remark 3.1 can be checked in a similar manner.

References

1. Keith Ball, *Ellipsoids of maximal volume in convex bodies*, Geom. Dedicata **41** (1992), no. 2, 241–250.
2. Marcel Berger, *Convexity*, Amer. Math. Monthly **97** (1990), no. 8, 650–678.
3. Fritz John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, N. Y., 1948, pp. 187–204.