Lecture 29 — 15 April, 2014

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1 Overview

In this lecture, we present the algorithm for fast support identification. We want to design measurements that allow us to quickly find an $S \subset [N]$ such that $S_0(k) \subset S$ for $\vec{x} \in \mathbb{C}^N$.

2 Notational review

Let $A \in \{0,1\}^{m \times N}$ and B_N be the N^{th} bit testing matrix. Let $\{\vec{b}_0, \vec{b}_1, \ldots, \vec{b}_{\lceil \log_2 N \rceil}\} \in \{0,1\}^N$ be the rows of B_N. Given $(A \otimes B_N)\vec{x}$ we also get $(A \otimes \vec{b}_i)\vec{x} \in \mathbb{C}^m$, $\forall i = 0 \ldots \lceil \log_2 N \rceil$. This means that we get $A\vec{x}$ as well as $(A(K,n) \otimes \vec{b}_i)\vec{x}$, $\forall n \in [N]$ and $\forall i = 0 \ldots \lceil \log_2 N \rceil$.

Note that $(A(K,n) \otimes \vec{b}_i) \in \{0,1\}^{K \times N}$ is exactly the matrix $A(K,n)$ with its l^{th}-column set to zero-vector if and only if $l \in [N]$ has a zero in its i^{th} bit when written in binary.

Example 1.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

Let $n \in [N]$ and assume that matrix A is (K,α)-coherent with $K > \frac{4\tilde{k}\alpha}{\varepsilon}$, where $\varepsilon \in (0,1)$ for sparsity $\tilde{k}$. Theorem 1 (Lecture 27) tells us that more than 1/2 of the $j \in [K]$ satisfy $(A(K,n)\vec{x})_j \in B(x_n, \delta)$, where

$$\delta := \frac{\varepsilon}{\tilde{k}} \left|\left| \vec{x} - \vec{x}_{S_0\left(\frac{\tilde{k}}{\varepsilon}\right)} \right|\right|_1 \qquad \tilde{k} \in [N], \forall \varepsilon \in (0,1) \tag{1}$$

Definition 1. *Given $\vec{x} \in \mathbb{C}^{\mathbb{N}}$, let $|\vec{x}| \in \mathbb{R}^{\mathbb{N}}$ be such that $|\vec{x}|_j := |\vec{x}_j|, \forall j \in [N]$.*

Note that δ is the same for both $\vec{x}$ and $|\vec{x}|$.

Now let's let $\vec{a}'_j \in \{0,1\}^N$ be the j^{th} row of $A(K,n)$ and suppose that

(i) $\langle \vec{a}'_j, |\vec{x}| \rangle \in B(|x_n|, \delta)$, and

(ii) $|x_n| > \delta$.

From Theorem 1 (Lecture 27) we know that more than 1/2 of the rows, $\vec{a}'_j$, of $A(K, n)$ will satisfy (i). Supposing that $|x_n| > \delta$ and that the i^{th} bit of n in binary is 1:

$$\begin{aligned}
\left|\langle \vec{a}'_j \otimes \vec{b}_i, \vec{x}\rangle\right| &\geq |x_n| - \sum_{\substack{l \in supp(\vec{a}'_j)\ s.t.\ l \neq n; \\ i^{th}\ bit\ of\ l=1}} |x_l| \\
&\geq \delta - \sum_{\substack{l \in supp(\vec{a}'_j)\ s.t.\ l \neq n; \\ i^{th}\ bit\ of\ l=1}} |x_l| \\
&\geq \sum_{\substack{l \in supp(\vec{a}'_j)\ s.t.\ l \neq n; \\ i^{th}\ bit\ of\ l=0}} |x_l| \\
&\geq \left|\langle \vec{a}'_j - \vec{a}'_j \otimes \vec{b}_i, \vec{x}\rangle\right|
\end{aligned} \tag{2}$$

Essentially the same argument shows that $\left|\langle \vec{a}'_j - \vec{a}'_j \otimes \vec{b}_i, \vec{x}\rangle\right| > \left|\langle \vec{a}'_j \otimes \vec{b}_i\rangle, \vec{x}\right|$, whenever the i^{th} bit of n is zero. We have now shown that the algorithm below will identify all $n \in [N]$ with $|x_n| > \delta$ more than $K/2$ times apeice.

Algortihm 1.

1. $S = \emptyset$
2. **For** $j \in [m]$
3. **For** $i = 0 \ldots \lceil \log_2 N \rceil - 1$
4. **If** $\left|\langle \vec{a}'_j \otimes \vec{b}_i, \vec{x}\rangle\right| > \left|\langle \vec{a}'_j - \vec{a}'_j \otimes \vec{b}_i, \vec{x}\rangle\right|$
   Set $n_i = 1$
5. **Else**
   Set $n_i = 0$
6. **End For**
7. Set $n = \sum_{i=0}^{\lceil \log_2 N \rceil - 1} n_i \cdot 2^i$ (translate from binary to decimal);
8. $S = S \cup \{n\}$
9. **End For**

It takes $O(m \log N)$ operations to go through steps 1 to 9. Also, we know that, e.g., $m = K^2$ is possible (from Lecture 26). Therefore, the total runtime of Algorithm 1 is generally sublinear in N. For example,

$$m = O\left(\frac{\tilde{k}^2 \log^3 N}{\varepsilon^2}\right) \ll N \tag{3}$$

works.

Measurements m can be randomized/reduced to get the total runtime of $O\left(\tilde{k}\log\left(\frac{N}{1-p}\right)\log\tilde{k}\right)$, which has the same accuracy as the deterministic variant with probability at least p.

It is true that $|S| \leq m$, but we also know that every $n \in [N]$ such that $|x_n| > \delta$ is recovered at least $K/2$ times. Therefore, $|S| = O(K)$, and we expect $S \supset S_0\left(\frac{2\tilde{k}}{\varepsilon}\right)$, which follows from the Lemma below.

Lemma 1. *Suppose that* $|x_n| > \delta$. *Then,* $n \in S_0\left(\frac{2\tilde{k}}{\varepsilon}\right)$. *As a result, Algorithm 1 finds all* $n \in S_0\left(\frac{2\tilde{k}}{\varepsilon}\right)$ *with* $|x_n| > \delta$.

Note that $n \in S_0\left(\frac{2\tilde{k}}{\varepsilon}\right)$ with $|x_n| \leq \delta$ are "OK to miss".

Next time we will use results from Lectures 28 and 29 to help construct Sparse Fast Fourier Transforms (SFFTs).