MTH 995-003: Intro to CS and Big Data	Spring 2014
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1 Combinatorial Properties of (k, α) Coherent Matrices

Let $A \in \{0,1\}^{m \times N}$ be a (k, α) coherent.

Definition 1. *$A(K,n) \in \{0,1\}^{K \times N}$ for a chosen $n \in [N]$ is the $K \times N$ submatrix of A created by selecting the first K rows of A with non-zero entries in n^{th} column of A.*

Example 1.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

is $(2,1)$ coherent. (Note that every column has exactly two $1's$ and the inner-product between any two columns in 1).

Then $A(2,3) \in \{0,1\}^{2 \times 8}$, where $K = 2$ and $n = 3$, is

$$A(2,3) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

(Recall: The column index starts from 0.)

Definition 2. *$A'(K,n) \in \{0,1\}^{K \times (N-1)}$ for $n \in [N]$ is the $A(K,n)$ sub-matrix of A with it's n^{th} column deleted.*

Example 2. *If A is as in the first example, then*

$$A'(2,3) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Lemma 1. *Suppose $A \in \{0,1\}^{m \times N}$ is (K, α)-coherent. Let $n \in [N], \tilde{k} \in [1, \frac{K}{\alpha}]$ and $\vec{x} \in \mathbb{C}^{N-1}$. Then at most $\tilde{k}\alpha$ of the entries of $A'(K,n)\vec{x} \in \mathbb{C}^K$ will have magnitude*

$$\geq \frac{\|\vec{x}\|_1}{\tilde{k}}$$

Proof. Let

$$B := \left\{ j \;\middle|\; |(A'(K,n)\vec{x})_j| \geq \frac{\|\vec{x}\|_1}{\tilde{k}} \right\}.$$

Then,

$$|B| \leq \frac{\tilde{k}}{\|\vec{x}\|_1} \|A'(K,n)\vec{x}\|_1 \leq \tilde{k} \|A'(K,n)\|_{1 \to 1}.$$

Bounding the operator norm of $A'(K,n)\vec{x}$ we get that

$$
\begin{aligned}
\|A'(K,n)\|_{1\to 1} &= \max_{l\in[N-1]} \|l^{th} \text{ column of } A'(K,n)\|_1 && (1)\\
&\le \max_{l\in[N]-\{n\}} \langle l^{th} \text{ column of } A, n^{th} \text{ column of } A\rangle && (2)\\
&\le \alpha. && (3)
\end{aligned}
$$

The result follows. □

Lemma 2. *Suppose $A \in \{0,1\}^{m\times N}$ is (K,α) coherent. Let $n \in [N], \tilde{k} \in [1, \frac{k}{\alpha}]$ and $S \subset [N]$ with $|S| \le \tilde{k}$. Let $\vec{x} \in \mathbb{C}^{N-1}$ then $A'(K,n)\vec{x}$ and $A'(K,n)\left(\vec{x} - \vec{x}_S\right)$ will differ in at most $\tilde{k}\alpha$ entries.*

Proof. Let $B \subset [K]$ defined by

$$B := \left\{ j \;\middle|\; (A'(K,n)\vec{x})_j \ne (A'(K,n)(\vec{x} - \vec{x}_S)_j \right\}.$$

Once can see that

$$|B| = \left| \left\{ j \;\middle|\; (A'(K,n)\vec{x}_S)_j \ne 0 \right\} \right|.$$

Let $\vec{q} \in \mathbb{C}^{N-1}$ be a vector of all $1's$. Note that:

- $|B| \le \left| \left\{ j \middle| (A'(K,n)\vec{q}_S)_j \ge 1 \right\} \right|$. Since $A'(K,n) \in \{0,1\}^{k\times(N-1)}$.
- Applying Lemma 1 with $\vec{x} = \vec{q}_S, \|\vec{x}\| = |S| \le \tilde{k}$ gives the result. □

Theorem 1. *Suppose A is (K,α)-coherent. Let $n \in [N], \tilde{k} \in [1, \frac{k}{\alpha}], \epsilon \in (0,1], c \in [2,\infty)$ and $\vec{x} \in \mathbb{C}^N$. If $K > \frac{c\tilde{k}\alpha}{\epsilon}$ then*

$$(A(K,n)\vec{x})_j \in B\left(x_n, \frac{\epsilon \left\| \vec{x} - \vec{x}_{S_0\left(\frac{\tilde{k}}{\epsilon}\right)} \right\|_1}{\tilde{k}}\right)$$

for more than $\frac{c-2}{c}K$ values of $j \in [K]$. Here $S_0(\tilde{k}) \subset [N]$ for a given $\vec{x} \in \mathbb{C}^N$ is the set of indexes $\left\{ j_1, j_2, \cdots, j_{\frac{\tilde{k}}{\alpha}} \right\}$ where $|x_{j_1}| \ge |x_{j_2}| \ge, \cdots, \ge \left| x_{j_{\frac{\tilde{k}}{\alpha}}} \right| \ge \ldots$.

Proof. Let $\vec{y} \in \mathbb{C}^{N-1}$ be $\vec{x}$ with x_n removed. i.e. $\vec{y} = (x_1, x_2, \cdots, x_{n-1}, x_{n+1}, \cdots, x_{N-1})$. Let $\vec{a_0}, \vec{a_1}, \cdots, \vec{a}_{N-1}$ be $\{0,1\}^{\tilde{K}}$ be the columns of $A(K,n)$. We have the following.

$$A(K,n)\vec{x} = x_n\vec{a_n} + A'(K,n)\vec{y} = x_n\vec{1} + A'(K,n)\vec{y}.$$

Applying Lemma 2, we see that at most $\tilde{k}\alpha/\epsilon$ entries of $A'(K,n)\vec{y}$ can differ from $A'(K,n)\left(\vec{y} - \vec{y}_{S_0\left(\frac{\tilde{k}}{\epsilon}\right)}\right)$. Of the remaining $K - \frac{\tilde{k}\alpha}{\epsilon}$ entries of $A'(K,n)\vec{y}$ at most $\frac{\tilde{k}\alpha}{\epsilon}$ entries can have magnitude

$$\ge \frac{\epsilon}{\tilde{k}} \left\| \vec{y} - \vec{y}_{S_0\left(\frac{\tilde{k}}{\epsilon}\right)} \right\|_1$$

by Lemma 1. Hence, at least $K - 2\frac{\tilde{k}\alpha}{\epsilon} > \frac{c-2}{c}K$ entries of $A'(K,n)\vec{y}$ will have magnitude

$$\leq \frac{\epsilon}{\tilde{k}} \left\| \vec{y} - \vec{y}_{S_0(\frac{k}{\epsilon})} \right\|_1 \leq \frac{\epsilon}{\tilde{k}} \left\| \vec{x} - \vec{x}_{S_0(\frac{k}{\epsilon})} \right\|_1 .$$

The result follows. $\square$

Note:

• Setting $c \geq 4$ in Theorem 1 tells us that the majority of $A(K,n)\vec{x}$ will be good (i.e., will lie within ball $B\left(x_n, \frac{\epsilon \left\| \vec{x} - \vec{x}_{S_0(\frac{k}{\alpha})} \right\|_1}{\tilde{k}}\right)$.

• *Recall from Example 1 of Lecture 26 that* $A \in \{0,1\}^{K^2 \times N}$ *matrices that are* $\left(K, \left\lfloor \frac{logN}{logK} \right\rfloor\right)$*-coherent exist. By Theorem 1, if we let*

$$K \geq 4\frac{\tilde{k}\log N}{\epsilon \ \log \tilde{k}}$$

then more than half of the $A'(K,n)\vec{x}$ *will estimate* x_n *well* $\forall n, \vec{x}$. *The number of rows is*

$$m = K^2 = O\left(\tilde{k}^2 \frac{\log^2 N}{\epsilon^2 \ \log^2 \tilde{k}}\right).$$