**MTH 995-003: Intro to CS and Big Data** Spring 2014

# Lecture 24 — Mar 27th, 2014

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## 1 Continuing from Lecture 23

Here we resume the proof of Lemma 3 from Lecture 23. Recall that we wanted to establish a bound

$$\mathsf{Vol}(B_r(\mathbf{p}) \cap \mathcal{M}) \geq f(r, \tau) \quad \text{for "most" } \mathbf{p} \in \mathcal{M}$$

for some function $f$ of $r$ and $\tau := \mathsf{reach}(\mathcal{M})$.

*Proof.* We saw that if

$$B_{r'}(\mathbf{p}) \cap T_{\mathbf{p}} \subseteq \Pi_{T_{\mathbf{p}}}(B_r(\mathbf{p}) \cap \mathcal{M}) \tag{1}$$

for some

$$r' \geq \sqrt{1 - \frac{r^2}{4\tau^2}} \cdot r,$$

then we would obtain the desired result.

Now, from Lemma 1 of Lecture 23, we know that $\Pi_{T_{\mathbf{p}}}$ is invertible on $B_r(\mathbf{p}) \cap \mathcal{M}$ for all $r \in [0, \frac{\tau}{4})$. This fact implies that $\Pi_{T_{\mathbf{p}}}(B_r(\mathbf{p}) \cap \mathcal{M})$ is open in $T_{\mathbf{p}}$. Thus, there exists $s \in \mathbb{R}^+$ such that

$$B_s(\mathbf{p}) \cap T_{\mathbf{p}} \subseteq \Pi_{T_{\mathbf{p}}}(B_r(\mathbf{p}) \cap \mathcal{M}). \tag{2}$$

Let $s^*$ be the supremum of all $s \in \mathbb{R}^+$ satisfying (2).

There is $\mathbf{y} \in \partial(B_{s^*}(\mathbf{p}) \cap T_{\mathbf{p}}) \cap \partial(\Pi_{T_{\mathbf{p}}}(B_r(\mathbf{p}) \cap \mathcal{M}))$. Set

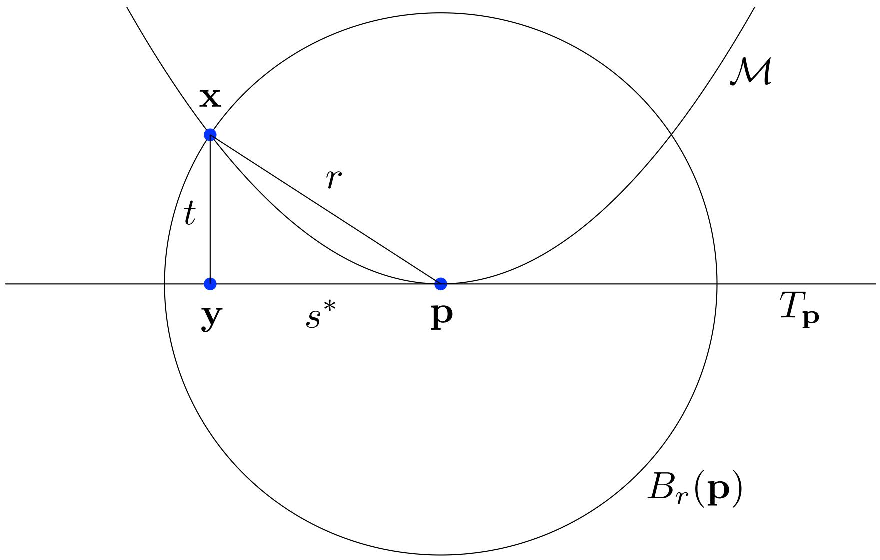
$$\mathbf{x} := \Pi_{T_{\mathbf{p}}}^{-1}(\mathbf{y}).$$

One can see that $\mathbf{x} \in \partial(B_r(\mathbf{p}) \cap \mathcal{M})$. Hence

$$\|\mathbf{x} - \mathbf{p}\|_2 = r$$

as long as, e.g., $B_r(\mathbf{p}) \cap \partial\mathcal{M} = \emptyset$. Finally, set

$$t := \|\mathbf{x} - \mathbf{y}\|_2.$$




Lemma 2 of Lecture 23 tells us

$$\angle \mathbf{ypx} \leq \arcsin\left(\frac{r}{2\tau}\right),$$

implying

$$\frac{t}{r} \leq \frac{r}{2\tau}.$$

And so

$$s^* \geq \sqrt{1 - \frac{r^2}{4\tau^2}} \cdot r,$$

meaning (1) follows by setting $r' := s^*$. □

The discussion at the end of Lecture 22 now gives us a covering number bound for at least the interior of $\mathcal{M}$.

**Definition 1.** *For a d-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^D$, the **r-interior** of $\mathcal{M}$ is*

$$\mathsf{int}_r(\mathcal{M}) \stackrel{\text{def}}{=} \{\mathbf{p} \in \mathcal{M} : B_r(\mathbf{p}) \cap \partial\mathcal{M} = \varnothing\}$$

We have proven the following result, which will help us prove Theorem 2, the desired manifold embedding result.

**Theorem 1.** *Let $\mathcal{M} \subseteq \mathbb{R}^D$ be a d-dimensional manifold with $\tau := \mathsf{reach}(\mathcal{M}) > 0$. Let $r \in [0, \frac{\tau}{4})$. Then the covering number will obey*

$$C_r\big(\mathsf{int}_r(\mathcal{M})\big) \leq \frac{\mathsf{Vol}_d(\mathcal{M})\Big(1 - \frac{r^2}{4\tau^2}\Big)^{\frac{-d}{2}} r^{-d}}{\mathsf{Vol}(\textit{unit ball in } \mathbb{R}^d)}.$$

## 2 The Johnson-Lindenstrauss Lemma and Manifold Embeddings

We wanted to show that a random matrix (in our case, one with subgaussian entries) will nearly isometrically embed any compact, $d$-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^D$ with positive reach, into $R^m$ such that $m << D$. The following theorem tells us precisely what this means.

**Theorem 2.** *Let $\mathcal{M} \subseteq \mathbb{R}^D$ be a d-dimensional, $\mathcal{C}^2$-manifold with $\mathsf{Vol}_d(\mathcal{M}) < \infty$, $\tau := \mathsf{reach}(\mathcal{M}) > 0$, and*

$$d\big(\mathbf{p}, \mathsf{int}_r(\mathcal{M})\big) \leq r \quad \textit{for all } r \in \Big[0, \frac{\tau}{4}\Big), \textit{ for all } \mathbf{p} \in \mathcal{M}.$$

*Let $\epsilon, \delta \in (0,1)$. Finally, let $A \in \mathbb{R}^{m \times D}$ with i.i.d. subgaussian entries (with parameter $c$). Then*

$$-\delta + (1-\epsilon)\|\mathbf{x}-\mathbf{y}\|_2 \leq \left\| \frac{1}{\sqrt{m}} A(\mathbf{x}-\mathbf{y}) \right\|_2 \leq (1+\epsilon)\|\mathbf{x}-\mathbf{y}\|_2 + \delta \quad \textit{for all } \mathbf{x}, \mathbf{y} \in \mathcal{M}$$

*with probability at least $p \in (0,1)$, provided*

$$m \geq \frac{(64c)(16c+1)}{\epsilon^2} \ln \Big( \frac{8}{1-p} C_{\tilde{r}}^2 \big(\mathsf{int}_{\tilde{r}}(\mathcal{M})\big) \Big),$$

*for*

$$\tilde{r} := \min \left\{ \sqrt{\frac{d}{D}} \frac{\delta}{18\sqrt{e}}, \frac{\tau}{4} \right\}.$$

- Theorem 1 tells us

$$C_{\tilde{r}}\big(\mathsf{int}_{\tilde{r}}(\mathcal{M})\big) \leq \frac{\mathsf{Vol}_d(\mathcal{M})}{\mathsf{Vol}(\text{unit ball in } \mathbb{R}^d)} \left(\frac{16}{15}\right)^{\frac{d}{2}} \max \left\{ \sqrt{\frac{D}{d}} \cdot \frac{18\sqrt{e}}{\delta\tau}, \frac{4}{\tau} \right\}^d.$$

Thus,

$$m \sim C' \frac{d}{\epsilon^2} \ln \left( \frac{\widetilde{C}}{\min\{\tau, 1\}(1-p)\delta} \cdot \frac{D}{d} \right)$$

for $D > 2d$ and constants $C'$ and $\widetilde{C}$ depending on $c$, and $\log(\mathsf{Vol}_d(\mathcal{M}))$, and assuming $d \ll D$.

With a bit more work, one can prove variants of Theorem 1 that make $m$ independent of $D$, specifically

$$m \sim C' \frac{d}{\epsilon^2} \ln \left( \frac{\widetilde{C} d}{\min\{\tau, 1\}(1-p)\delta} \right).$$

With a substantial amount of work, one can prove

$$m \sim C' \frac{d}{\epsilon^2} \ln \left( \frac{\widetilde{C} d}{\min\{\tau, 1\}(1-p)} \right).$$

For these results, see [1, 2], respectively.

Let's now prove Theorem 2.

*Proof.* Let $C \subseteq \mathcal{M}$ be a minimal $\widetilde{r}$-cover of $\mathsf{int}_{\widetilde{r}}(\mathcal{M})$. Note that $C$ is also a $2\widetilde{r}$-cover of $\mathcal{M} \subseteq \mathbb{R}^D$.

Theorem 1 of Lecture 14 guarantees that $\widetilde{A} := \frac{1}{\sqrt{m}} A$, with $m$ as above, will satisfy

$$(1-\epsilon) \leq \sqrt{1-\epsilon} \leq \frac{\|A(\mathbf{p}-\mathbf{q})\|_2}{\|\mathbf{p}-\mathbf{q}\|_2} \leq \sqrt{1+\epsilon} \leq 1+\epsilon \quad \text{for all } \mathbf{p}, \mathbf{q} \in C \tag{3}$$

with probability at least $1 - \frac{1-p}{2}$.

Now, Theorem 1 of Lecture 15 guarantees that $\widetilde{A}$ also has the RIP of order $d$ for $\epsilon < 1$, with probability at least $1 - \frac{1-p}{2}$. That is, $\epsilon_d(\widetilde{A}) \in (0,1)$, implying

$$\sigma_1(\widetilde{A}) \leq 2\sqrt{2}\sqrt{\frac{D}{d}} \tag{4}$$

by Lemma 2 of Lecture 16. The union bound implies that (3) and (4) hold simultaneously with probability at least $p$.

Thus,

$$\begin{aligned} \|\widetilde{A}(\mathbf{x}-\mathbf{y})\|_2 &\leq \|\widetilde{A}(\mathbf{x}-\mathbf{p_x})\|_2 + \|\widetilde{A}(\mathbf{p_x}-\mathbf{p_y})\|_2 + \|\widetilde{A}(\mathbf{p_y}-\mathbf{y})\|_2 \\ &\leq 2\sqrt{2}\sqrt{\frac{D}{d}}\left(\|\mathbf{x}-\mathbf{p_x}\|_2 + \|\mathbf{p_y}-\mathbf{y}\|_2\right) + (1+\epsilon)\|\mathbf{p_x}-\mathbf{p_y}\|_2, \end{aligned}$$

where $\mathbf{p_x}$ and $\mathbf{p_y}$ are the closest points in $C$ to $\mathbf{x}$ and $\mathbf{y}$, respectively. That is,

$$\mathbf{p_x} = \arg\min_{\mathbf{p}\in C} \|\mathbf{p}-\mathbf{x}\|_2 \quad \text{and} \quad \mathbf{p_y} = \arg\min_{\mathbf{p}\in C} \|\mathbf{p}-\mathbf{y}\|_2.$$

As $C$ is a $2\widetilde{r}$-cover, $\|\mathbf{x}-\mathbf{p_x}\|_2$ and $\|\mathbf{p_y}-\mathbf{y}\|_2$ are bounded from above by $2\widetilde{r}$, while an additional application of the triangle inequality gives $\|\mathbf{p_x}-\mathbf{p_y}\|_2 \leq \|\mathbf{x}-\mathbf{y}\|_2 + 4\widetilde{r}$. When used above, these estimates yield

$$\|\widetilde{A}(\mathbf{x}-\mathbf{y})\|_2 \leq \frac{6\delta}{9} + (1+\epsilon)\|\mathbf{x}-\mathbf{y}\|_2,$$

giving the desired upper bound. An analogous argument gives the desired lower bound. □

# References


[1] Mark A. Iwen, Mauro Maggioni. Approximation of Points on Low-Dimensional Manifolds Via Random Linear Projections. *J. CoRR*, 1204.3337, 2012.

[2] Armin Eftekhari, Michael B. Wakin. New Analysis of Manifold Embeddings and Signal Recovery from Compressive Measurements. *J. CoRR* , 1306.4748, 2013.