MTH 995-003: Intro to CS and Big Data	Spring 2014
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1 Overview

In the last lecture we introduced two important classes of random variables (RVs): sub-exponentials and sub-gausians. Recall that X is a sub-gaussian RV if $\exists\, \beta, \kappa > 0$ such that

$$\mathbb{P}\left[|X| \geq t\right] \leq \beta e^{-\kappa t^2} \qquad \forall t > 0 \tag{1}$$

This lecture provides a few more important results regarding the characterization of sub-gaussians. We will begin by bounding the absolute moments of sub-gaussian RVs in terms of their parameters β and κ.

2 Two Useful Lemmas Concerning Moments and MGFs

Lemma 1. *If X is a subgaussian random variable with parameters $\beta > 0$ and $\kappa > 0$, then we can bound its moments such that*

$$\left(\mathbb{E}\left[|X|^p\right]\right)^{\frac{1}{p}} \leq \kappa^{-\frac{1}{2}} \beta^{\frac{1}{p}} p^{\frac{1}{2}} \qquad \forall p \geq 1 \tag{2}$$

Proof:

Inequality (16) from Lecture 11 tells us that

$$\mathbb{E}\left[|X|^p\right] = p \int_0^\infty \mathbb{P}\left[|X| \geq t\right] t^{p-1} dt \tag{3}$$

After the change of variables $t \to \frac{u}{\sqrt{2\kappa}}$ we will have

$$\mathbb{E}\left[|X|^p\right] = \frac{p}{(2\kappa)^{\frac{p}{2}}} \int_0^\infty \mathbb{P}\left[|X| \geq \frac{u}{\sqrt{2\kappa}}\right] u^{p-1} du \leq \frac{\beta p}{(2\kappa)^{\frac{p}{2}}} \int_0^\infty e^{-\frac{u^2}{2}} u^{p-1} du \tag{4}$$

where we have used the fact that X is a subgaussian RV. After the second change of variables $x \to -\frac{u^2}{2}$:

$$\mathbb{E}\left[|X|^p\right] \leq \frac{\beta p}{2\kappa^{\frac{p}{2}}} \int_0^\infty e^{-x} x^{\frac{p}{2}-1} dX \tag{5}$$

Note that the integral above is the Gamma-function evaluated at $p/2$. Thus,

$$\mathbb{E}\left[|X|^p\right] \leq \frac{\beta p}{2\kappa^{\frac{p}{2}}} \Gamma\left(\frac{p}{2}\right) \tag{6}$$

Applying Sterling's formula to (6) yields

$$\mathbb{E}\left[|X|^p\right] \leq \sqrt{\frac{\pi}{2}} \cdot \frac{p\beta}{\kappa^{\frac{p}{2}}} \cdot \left(\frac{p}{2}\right)^{\frac{p-1}{2}} e^{\frac{-p}{2}} e^{\frac{1}{6p}} = \kappa^{-\frac{p}{2}} \beta p^{\frac{p}{2}} \left[\sqrt{\frac{p\pi}{(2e)^p}} \cdot e^{\frac{1}{6p}}\right] \leq 0.9 \cdot \kappa^{-\frac{p}{2}} \beta p^{\frac{p}{2}} \tag{7}$$

$\forall p \geq 1$. □

Lemma 2. *If X is a subgaussian random variable with parameters $\beta > 0$ and $\kappa > 0$, then $\exists\, c \in (0, \kappa)$ and $\tilde{c} \geq 1 + \frac{\beta c \kappa^{-1}}{1 - c\kappa^{-1}}$ such that*

$$\mathbb{E}\left[e^{cX^2}\right] \leq \tilde{c}. \tag{8}$$

Proof:

The moment estimate, (6), from the proof of Lemma 1 tells us that

$$\mathbb{E}\left[|X|^{2n}\right] \leq \beta\kappa^{-n} n\Gamma(n) = \beta\kappa^{-n} n! \tag{9}$$

By Fubini and Taylor's Theorem (once again...) we get that

$$\mathbb{E}\left[e^{cX^2}\right] = \int_0^\infty \sum_{n=0}^\infty \frac{c^n X^{2n}}{n!} d\mathbb{P} \leq \sum_{n=0}^\infty \frac{c^n \mathbb{E}[|X|^{2n}]}{n!}. \tag{10}$$

The assumption $c \in (0, \kappa)$ ensures convergence. Applying (9) and then summing up the series yields

$$\mathbb{E}\left[e^{cX^2}\right] \leq 1 + \beta \sum_{n=1}^\infty c^n \kappa^{-n} \leq 1 + \frac{\beta c \kappa^{-1}}{1 - c\kappa^{-1}} \leq \tilde{c}. \tag{11}$$

□

3 A Characterization of Subgaussian Random Variables

Theorem 1 (see [1], p.193)**.** *Let X be a random variable, then:*

1. *If X is a sub-gaussian RV with parameters $\beta > 0$, $\kappa > 0$, and has $\mathbb{E}[X] = 0$, then $\forall c \in \mathbb{R}^+$ with*

$$c > \max\left\{\frac{1}{2\kappa} + \frac{4e^2}{\kappa} \ln(1+\beta),\ \frac{\sqrt{2}\beta e^2}{\kappa\sqrt{\pi}}\right\} \tag{12}$$

   we have

$$\mathbb{E}\left[e^{\Theta X}\right] \leq e^{c\Theta^2} \qquad \forall \Theta \in \mathbb{R} \tag{13}$$

2. *If property (13) holds for $c \in \mathbb{R}$, then $\mathbb{E}[X] = 0$ and X is a sub-gaussian RV with parameters $\beta = 2$ and $\kappa = \frac{1}{4c}$.*

Proof of part (2):

Let $\Theta > 0$ and $t > 0$. Then

$$\mathbb{P}\left[X \geq t\right] = \mathbb{P}\left[e^{\Theta X} \geq e^{\Theta t}\right] \leq e^{-\Theta t} \cdot \mathbb{E}\left[e^{\Theta X}\right] \tag{14}$$

by Markov's inequality. Using our assumption (13):

$$\mathbb{P}\left[X \geq t\right] \leq e^{c\Theta^2 - \Theta t} \tag{15}$$

After minimizimg over Θ, the optimal value can be shown to be $\Theta = \frac{t}{2c}$. Hence,

$$\mathbb{P}\left[X \geq t\right] \leq e^{-t^2/4c}. \tag{16}$$

Similarly,

$$\mathbb{P}\left[-X \geq t\right] \leq e^{-t^2/4c}. \tag{17}$$

Applying the union bound, we have

$$\mathbb{P}\left[|X| \geq t\right] \leq 2e^{-t^2/4c} \tag{18}$$

This gives us subgaussianity as needed, now we shall show that $\mathbb{E}\left[X\right] = 0$. Note that $1 + x \leq e^x$ for $\forall x$. Thus $1 + \Theta X \leq e^{\Theta X}, \ \ \forall \Theta \in \mathbb{R}$, and so

$$\mathbb{E}\left[1 + \Theta X\right] \leq \mathbb{E}\left[e^{\Theta X}\right] \tag{19}$$

Hence, by the inequality (13)

$$1 + \mathbb{E}\left[\Theta X\right] \leq e^{c\Theta^2} \tag{20}$$

If Θ is sufficiently small

$$1 + \Theta \mathbb{E}\left[X\right] \leq 1 + 2c\Theta^2 \qquad \forall \Theta \in \left(-\frac{1}{\sqrt{c}}, \frac{1}{\sqrt{c}}\right) \tag{21}$$

$$\left|\mathbb{E}\left[X\right]\right| \leq 2c\Theta \qquad \forall \Theta \in \left[0, \frac{1}{\sqrt{2c}}\right) \tag{22}$$

It follows that $\mathbb{E}\left[X\right] = 0$.

Proof of part (1):

Let us expand $\mathbb{E}\left[e^{\Theta X}\right]$ using Taylor's Theorem, Fubini's Theorem, and the assumption that $\mathbb{E}\left[X\right] = 0$:

$$\mathbb{E}\left[e^{\Theta X}\right] \leq 1 + \sum_{n=2}^{\infty} \frac{\Theta^n}{n!} \mathbb{E}\left[|X|^n\right] \leq 1 + \sum_{n=2}^{\infty} \frac{|\Theta|^n}{n!} \mathbb{E}\left[|X|^n\right] \tag{23}$$

Using Sterling's approximation of $n!$, Lemma 1, and assuming that $|\Theta| \leq \Theta_0$ for Θ_0 sufficiently small, we have

$$\mathbb{E}\left[e^{\Theta X}\right] \leq 1 + \sum_{n=2}^{\infty} \frac{|\Theta|^n \kappa^{-\frac{n}{2}} \beta n^{\frac{n}{2}}}{\sqrt{2\pi} n^n e^{-n}} \leq 1 + \frac{\beta \Theta^2 e^2}{\sqrt{2\pi}\kappa} \sum_{n=0}^{\infty} \left(\Theta_0 \kappa^{-\frac{1}{2}} e\right)^n = 1 + \frac{\Theta^2 \beta e^2}{\sqrt{2\pi}\kappa} \cdot \frac{1}{1 - \Theta_0 e \kappa^{-\frac{1}{2}}} \tag{24}$$

provided that $\Theta_0 < \frac{\sqrt{\kappa}}{e}$. Setting $\Theta_0 = \frac{\sqrt{\kappa}}{2e}$ results in

$$\mathbb{E}\left[e^{\Theta X}\right] \leq \exp\left\{\frac{\Theta^2\sqrt{2}\beta e^2}{\sqrt{\pi}\kappa}\right\}. \tag{25}$$

The exponent in (25) gives us one of our lower bounds on c we can achieve.

But what happens if $|\Theta| > \Theta_0$? Note that

$$\Theta X - \tilde{c}\Theta^2 = -\left(\sqrt{\tilde{c}}\,|\Theta| - \frac{X}{2\sqrt{\tilde{c}}}\right)^2 + \frac{X^2}{4\tilde{c}} \leq \frac{X^2}{4\tilde{c}} \tag{26}$$

$\forall$ realizations of X and $\forall \tilde{c} \in \mathbb{R}^+$. Let the constant from Lemma 2 be $c_2 \in (0, \kappa)$ such that

$$\mathbb{E}\left[e^{c_2 X^2}\right] \leq c' \tag{27}$$

by Lemma 2. Then take $\tilde{c} = \frac{1}{4c_2} = \frac{1}{2\kappa}$. Now (26) and Lemma 2 imply that

$$\mathbb{E}\left[\exp\left\{\Theta X - \frac{\Theta^2}{2\kappa}\right\}\right] \leq \mathbb{E}\left[\exp\left\{\frac{X^2\kappa}{2}\right\}\right] \leq c' = 1 + \beta. \tag{28}$$

Hence,

$$\mathbb{E}\left[e^{\Theta X}\right] \leq (1+\beta)\exp\left\{\frac{\Theta^2}{2\kappa}\right\} = (1+\beta)\exp\left\{-\ln(1+\beta)\frac{\Theta^2}{\Theta_0^2}\right\} \cdot \exp\left\{\frac{\Theta^2}{2\kappa} + \ln(1+\beta)\frac{\Theta^2}{\Theta_0^2}\right\} \tag{29}$$

Since $|\Theta| > \Theta_0$ we now can see that

$$\mathbb{E}\left[e^{\Theta X}\right] \leq \exp\left\{\Theta^2 \cdot \left(\frac{1}{2\kappa} + \frac{\ln(1+\beta)}{\Theta_0^2}\right)\right\}. \tag{30}$$

The exponent in (30) gives us our other lower bound on c. □

In the next lectures we will show that parameter c is quite important, because it is closely related to the size (and sparsity!) of random sampling matrices used in Compressive Sensing.

4 Homework 3

3). Let X be a random variable with the PDF

$$f(x) = p \cdot \delta(x) + \frac{(1-p)^{\frac{3}{2}}}{\sqrt{2\pi}} \exp\left\{\frac{-x^2(1-p)}{2}\right\} \quad for \quad p \in (0,1) \tag{31}$$

Show that X is a subgaussian random variable with $\mathbb{E}[X] = 0$, $\mathbb{VAR}[X] = 1$ and $c = \frac{1}{2(1-p)}$.

4). Let X be a random variable with the PDF

$$f(x) = p \cdot \delta(x) + \left(\frac{1-p}{2}\right)\left[\delta\left(x - \frac{1}{\sqrt{1-p}}\right) + \delta\left(x + \frac{1}{\sqrt{1-p}}\right)\right] \quad for \quad p \in (0,1) \tag{32}$$

Note that in this case $X = 0$ with probability p, and $X = \pm\frac{1}{\sqrt{1-p}}$ each with probability $\frac{1-p}{2}$. Show that X is a subgaussian random variable with $\mathbb{E}[X] = 0$, $\mathbb{VAR}[X] = 1$ and $c = \frac{1}{1-p}$. For what values of p can you achieve $c = \frac{1}{\sqrt{1-p}}$?

References

[1] Simon Foucart and Holger Rauhut. A Mathematical Introduction to Compressive Sensing. Birkhauser Basel, 2013.