MTH 995-001: Intro to CS and Big Data	Spring 2015
Lecture 15 — March 5, 2015	
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1 Hoeffding's Inequality and Corollary

This adds to **Lecture 11** from Spring 2014, which details Cramer's Theorem used below.

Theorem 1 (Hoeffding's Inequality)**.** *Let $X_1, X_2, \ldots, X_M$ be independent random variables such that $\mathbb{E}[X_l] = 0$ and $|X_l| \leq B_l \ \ \forall\, l \in [M]$. Then, $\forall\, t > 0$*

$$\mathbb{P}\left(\sum_{l=1}^{M} X_l \geq t\right) \leq \exp\left(-\frac{t^2}{2\sum_{l=1}^{M} B_l}\right) \tag{1}$$

and

$$\mathbb{P}\left(\left|\sum_{l=1}^{M} X_l\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sum_{l=1}^{M} B_l}\right) \tag{2}$$

Proof: We will estimate $\mathbb{E}[\exp(\theta X_l)]$ and then use Cramer's theorem.

$$X_l = \widetilde{t}\,(-B_l) + \left(1-\widetilde{t}\right) B_l \quad \text{where} \quad \widetilde{t} = \frac{B_l - X_l}{2B_l} \text{ is random}$$

Since $\exp(\theta X_l)$ is a convex function, we have

$$\begin{aligned}\exp(\theta X_l) &\leq \widetilde{t}\exp(-\theta B_l) + \left(1-\widetilde{t}\right)\exp(\theta B_l)\\ &= \left(\frac{B_l - X_l}{2B_l}\right)\exp(-\theta B_l) + \left(\frac{B_l + X_l}{2B_l}\right)\exp(\theta B_l)\end{aligned}$$

Taking the expectation of both sides yields

$$\begin{aligned}\mathbb{E}[\exp(\theta X_l)] &\leq \frac{1}{2}\exp(-\theta B_l) + \frac{1}{2}\exp(\theta B_l) = \sum_{k=0}^{\infty}\frac{(\theta B_l)^{2k}}{(2k)!}\\ &\leq \sum_{k=0}^{\infty}\frac{(\theta B_l)^{2k}}{2^k k!} = \exp\left(\frac{\theta^2 B_l^2}{2}\right)\end{aligned}$$

If we apply Cramer's theorem with $\theta = \frac{\widetilde{t}}{\sum_{l=1}^{M} B_l^2}$, rearrange with alegbra, we get the result in Equations 1 and 2. $\square$

Corollary. Let $a \in \mathbb{R}^N$ and

$$X_l = \begin{cases} +1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

Let random vector $\vec{b}$ have entries $b_l = X_l$ and be I.I.D. Then, $\forall u > 0$ we have

$$\begin{aligned} \mathbb{P}\left[\left|\left\langle \vec{a}, \vec{b} \right\rangle\right| \geq \|\vec{a}\|_2 \, u\right] &= \mathbb{P}\left[\left|\sum_{l=1}^{N} a_l X_l\right| \geq \|\vec{a}\|_2 \, u\right] \\ &\leq 2 \exp\left(-\frac{u^2}{2}\right) && \text{By Hoeffding's Inequality} \end{aligned}$$