A Non-sparse Tutorial on Sparse FFTs

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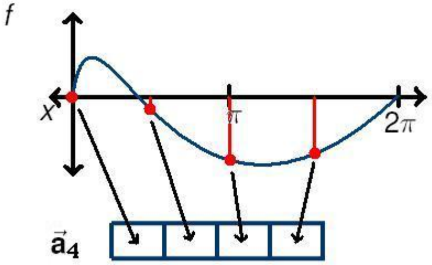
Problem Setup

Recover $f : [0, 2\pi] \mapsto \mathbb{C}$ consisting of k trigonometric terms

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathbb{e}^{x \cdot \omega_j \cdot \mathbb{i}}, \; \Omega = \{\omega_1, \dots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$



- Approximate $\{(\omega_j, C_j) \mid 1 \leq j \leq k\}$ using only $\vec{a}_N$

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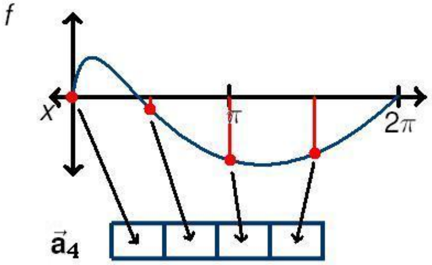

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A Woefully Incomplete History of "Fast" Sparse FFTs

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- The Fast Fourier Transform (FFT) [CT'65] can approximate (ω_j, C_j), $1 \leq j \leq k$, in $O(N \log N)$-time. Efficient FFT implementations that minimize the hidden constants have been developed (e.g., FFTW [FJ' 05]).
- Mansour [M'95]; Akavia, Goldwasser, Safra [AGS' 03]; Gilbert, Guha, Indyk, Muthukrishnan, Strauss [GGIMS' 02] & [GMS' 05]; I., Segal [I'13] & [SI'12]; Hassanieh, Indyk, Katabi, Price [HIKPs'12] & [HIKPst'12]; ... $O(k \log^c N)$-time

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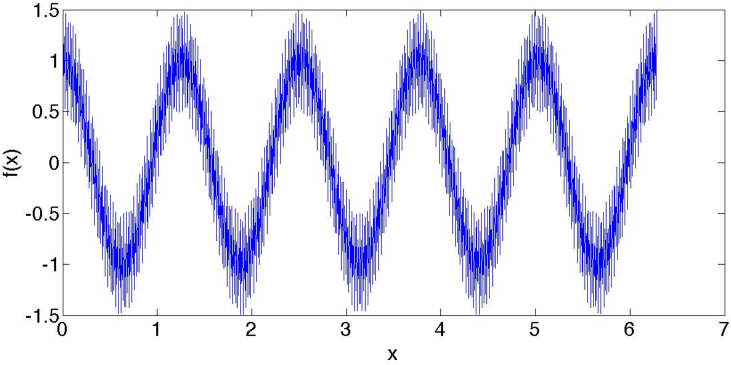
Example: $\cos(5x) + .5\cos(400x)$



- $f(x) = (1/4)\mathbb{e}^{-400x\cdot\mathbb{i}} + (1/2)\mathbb{e}^{-5x\cdot\mathbb{i}} + (1/2)\mathbb{e}^{5x\cdot\mathbb{i}} + (1/4)\mathbb{e}^{400x\cdot\mathbb{i}}$
- $\Omega = \{-400, -5, 5, 400\}$
- $C_1 = C_4 = 1/4$, and $C_2 = C_3 = 1/2$

Sparse Fourier Recovery

Suppose $f : [0, 2\pi]^D \mapsto \mathbb{C}$ has $\hat{f} \in \ell^1$. Let $N, D, d, \epsilon^{-1} \in \mathbb{N}$. Then, a simple algorithm, $\mathcal{A}$, can output an $\mathcal{A}(f) \in \mathbb{C}^{N^D}$ satisfying

$$\left\| \vec{\hat{f}} - \mathcal{A}(f) \right\|_2 \leq \left\| \vec{\hat{f}} - \vec{\hat{f}}_d^{\,\mathrm{opt}} \right\|_2 + \frac{\epsilon \cdot \left\| \vec{\hat{f}} - \vec{\hat{f}}_{(d/\epsilon)}^{\,\mathrm{opt}} \right\|_1}{\sqrt{d}} + 22\sqrt{d} \cdot \left\| \hat{f} - \vec{\hat{f}} \right\|_1 .$$

The runtime as well as the number of function evaluations of f are both

$$O\left(\frac{d^2 \cdot D^4 \cdot \log^4 N}{\epsilon^2 \cdot \log D} \right).$$

- $\vec{\hat{f}} \in \mathbb{C}^{N^D}$ consists of $\hat{f}$ for $\vec{\omega} \in \mathbb{Z}^D$ with $\|\vec{\omega}\|_\infty \leq N/2$
- $\vec{\hat{f}}_d^{\,\mathrm{opt}} \in \mathbb{C}^{N^D}$ is a best d-sparse approximation to $\vec{\hat{f}}$

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- A randomized result achieves the same bounds w.h.p. using $O\left(\frac{d \cdot D^4 \cdot \log^5 N}{\epsilon \cdot \log D}\right)$ samples and runtime.
- The full FFT uses $O\left(N^D \cdot D \cdot \log N\right)$ operations

Four Step Approach

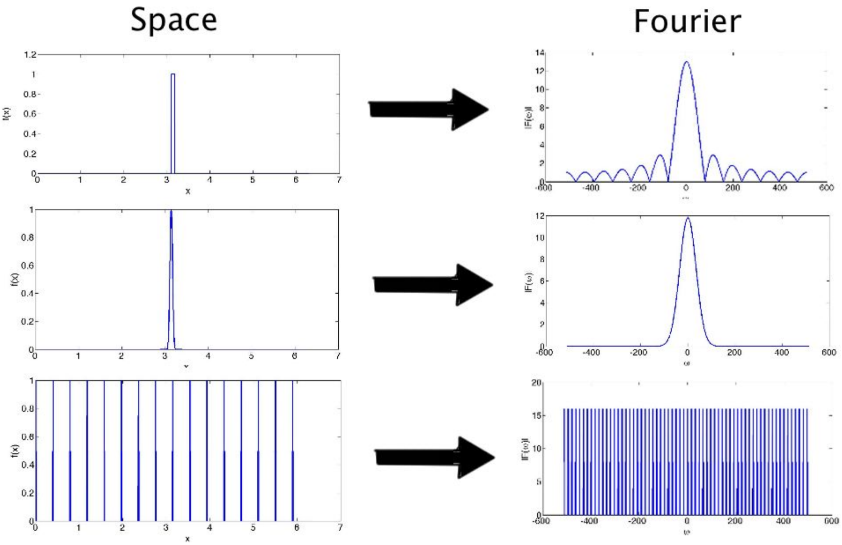
Approximate $\left\{(\omega_j, C_j) \,\middle|\, 1 \le j \le k\right\}$ by sampling

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathbb{e}^{x \cdot \omega_j \cdot \mathbb{i}}, \; \Omega = \{\omega_1, \dots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

A Sparse Fourier Transform will...

1. Try to isolate each frequency, $\omega_j \in \Omega$, in some
$$f_j(x) = C_j' \cdot \mathbb{e}^{x \cdot \omega_j \cdot \mathbb{i}} \; + \; \epsilon(x)$$
2. $\tilde{\Omega} \leftarrow$ Use $f_j(x)$ to learn all $\omega_j \in \Omega$
3. $\tilde{C}_j \leftarrow$ Estimate C_j for each $\omega_j \in \tilde{\Omega}$
4. Repeat on $f - \sum_{\omega_j \in \tilde{\Omega}} \tilde{C}_j \cdot \mathbb{e}^{x \cdot \omega_j \cdot \mathbb{i}}$, or not...

Design Decision #1: Pick a Filter



Previous Choices

- (Indicator function,Dirichlet) Pair: [GGIMS' 02] & [GMS' 05]
- (Spike Train,Spike Train) Pair: [I'13] & [SI'12]
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We'll use a regular Gaussian today

[1] Also consider Dolph-Chebyshev window function...

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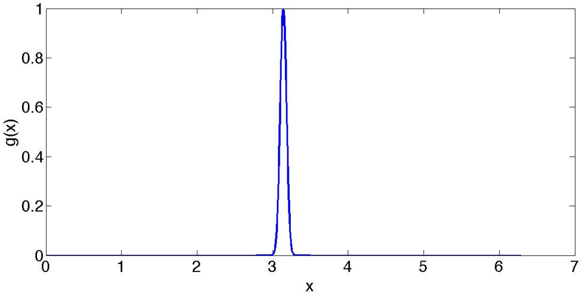
Gaussian with "Small Support" in Space

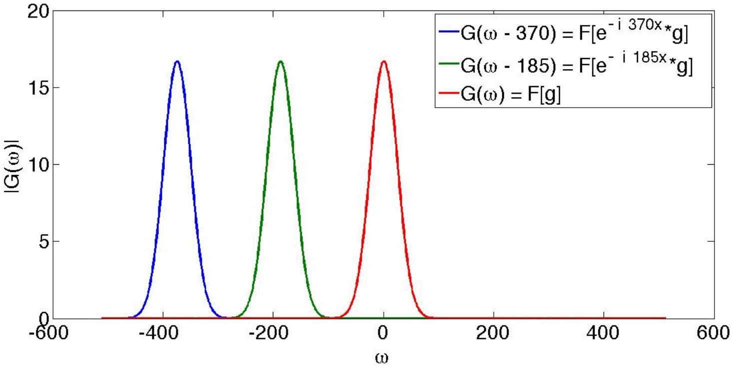


- Supports fast approximate convolutions: $\text{Conv}[g, f](j\Delta x)$ is

$$\sum_{h=0}^{N-1} g(h\Delta x) f\left((j-h)\Delta x\right) \approx \sum_{h=N/2-c}^{N/2+c} g(h\Delta x) f\left((j-h)\Delta x\right).$$

- $\Delta x = 2\pi/N$, c small

Gaussian has “Large Support” in Fourier



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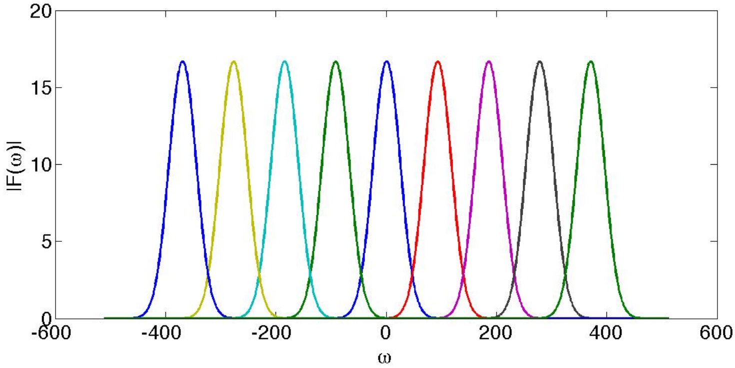

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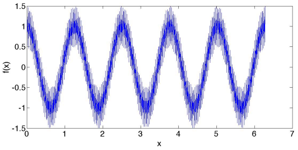
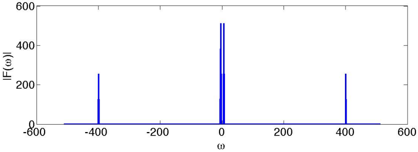
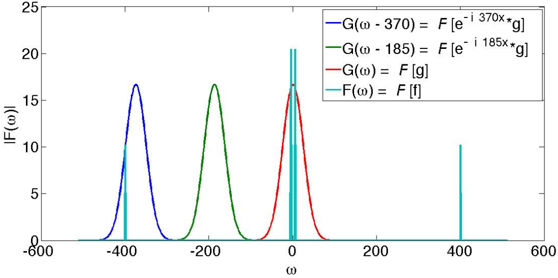
Example: Convolutions Bin Fourier Spectrum





- $\mathcal{F}\left[Conv[g,f](x)\right](\omega) = \mathcal{F}[g](\omega) * \mathcal{F}[f](\omega)$
- Convolving allows us to select parts of *f*'s spectrum



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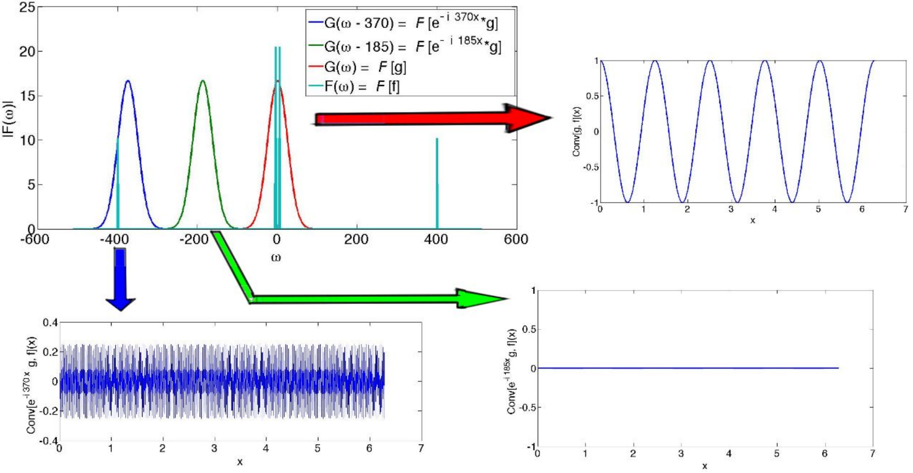


Binning Summary

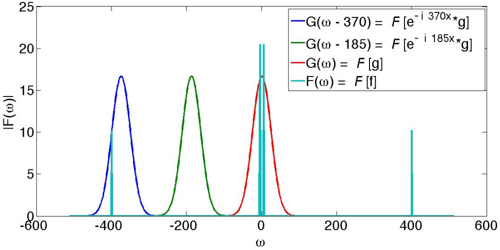
1. Large support in Fourier $\implies$ Need few modulations of g to bin

$$\mathbb{e}^{-\mathbb{i}2ax}g(x),\ \mathbb{e}^{-\mathbb{i}ax}g(x),\ g(x),\ \mathbb{e}^{\mathbb{i}ax}g(x),\ \mathbb{e}^{\mathbb{i}2ax}g(x)$$

2. Small Support in Space $\implies$ Need few samples for convolutions

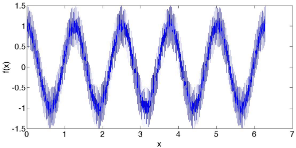
$$\mathrm{Conv}[\mathbb{e}^{-\mathbb{i}ax}g, f](j\Delta x) \approx \sum_{h=\frac{N}{2}-c}^{\frac{N}{2}+c} \mathbb{e}^{-\mathbb{i}ah\Delta x} g(h\Delta x) f\left((j-h)\Delta x\right),\ c \text{ small}$$

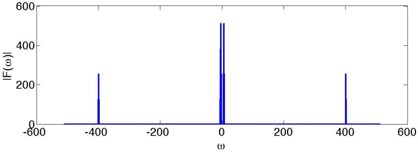
3. Problem: Two frequencies can be binned in the same bucket



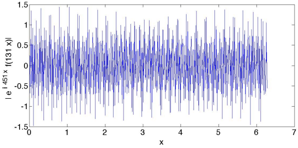
Shift and Spread the Spectrum of f

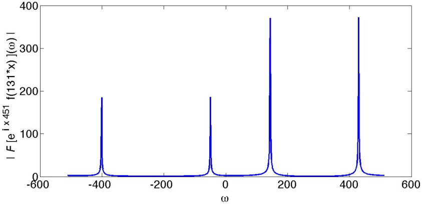
- f $\qquad$ $\mathcal{F}[f](\omega)$



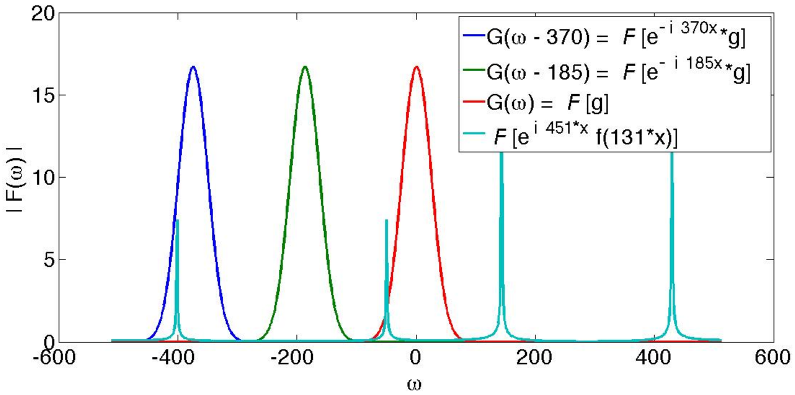


- $e^{i451x} f(131 * x)$ $\qquad$ $\mathcal{F}\left[e^{i451x} f(131 * x)\right](\omega)$





Frequency Isolation



- We have isolated one of the previously collided frequencies in

$$\mathrm{Conv}[\mathbb{e}^{-\mathbb{i}370x}g(x),\ \mathbb{e}^{\mathbb{i}451x}f(131x)](x)$$

Frequency Isolation Summary

1. Choose filter g with small support in space, large support in Fourier

2. Randomly select dilation and modulation pairs, $(d_l, m_l) \in \mathbb{Z}^2$

3. Each energetic frequency in f, $\omega_j \in \Omega$, will have a proxy isolated in

$$\mathrm{Conv}[\mathbb{e}^{-\mathbb{i}nax} g(x),\ \mathbb{e}^{\mathbb{i}m_l x} f(d_l x)](x)$$

   for some n, m_l, d_l triple with high probability.

4. Analyzing probability of isolation is akin to considering tossing balls (frequencies of f) into bins (pass regions of modulated filter)

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Design Decision #2: Frequency Identification

Frequency Isolated in a Convolution

$$f_j(x) := \mathrm{Conv}[\mathbb{e}^{-\mathbb{i} n_j a x} g(x),\ \mathbb{e}^{\mathbb{i} m_{l_j} x} f(d_{l_j} x)](x) = C_j' \cdot \mathbb{e}^{x \cdot \omega_j' \cdot \mathbb{i}} + \epsilon(x)$$

1. Compute the phase of

$$\frac{f_j(h_1 \Delta x)}{f_j(h_1 \Delta x + \pi)} \approx \mathbb{e}^{\pi \mathbb{i} \cdot \omega_j'}$$

2. Perform a modified binary search for ω_j'. A variety of methods exist for making decisions about the set of frequencies ω_j' belongs to at each stage of the search...

Identification Example: One Nonzero Entry

- $M \in \{0,1\}^{5\times 6}$, $\hat{f}_j \in \mathbb{C}^6$ contains 1 nonzero entry.

$$\begin{matrix} \equiv 0 \bmod 2 \\ \equiv 1 \bmod 2 \\ \equiv 0 \bmod 3 \\ \equiv 1 \bmod 3 \\ \equiv 2 \bmod 3 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 3.5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- Reconstruct entry index via Chinese Remainder Theorem
- Two estimates of the entry's value

SAVED ONE LINEAR TEST!

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- We only utilize 4 samples
- Computed Efficiently using 2 FFTs
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- Two estimates of nonzero Fourier coefficient

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$$\begin{pmatrix} \sqrt{\frac{3}{2}} & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ * & 0 & * & 0 & * & 0 \\ * & 0 & * & 0 & * & 0 \\ * & 0 & * & 0 & * & 0 \end{pmatrix} \cdot \left(\mathcal{F}^{-1}_{6\times 6} \begin{pmatrix} 0 \\ 0 \\ 3.5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 3.5 \\ 0 \\ 0 \\ 0 \\ 3.5 \end{pmatrix}$$

- We only utilize 4 samples
- Computed Efficiently using 2 FFTs
- Reconstruct frequency index via Chinese Remainder Theorem
- Two estimates of nonzero Fourier coefficient

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Identification Example: One Fourier Coefficient

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Identification Example: One Fourier Coefficient

$$\begin{pmatrix} \sqrt{3} \cdot \mathcal{F}_{2\times 2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\ \sqrt{2} \cdot \mathcal{F}_{3\times 3} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{pmatrix} \cdot \left(\mathcal{F}^{-1}_{6\times 6} \begin{pmatrix} 0 \\ 0 \\ 3.5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 3.5 \\ 0 \\ 0 \\ 0 \\ 3.5 \end{pmatrix}$$

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Identification Example: One Fourier Coefficient

$$\left(\begin{array}{c} \sqrt{3} \cdot \mathcal{F}_{2\times 2} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\ \sqrt{2} \cdot \mathcal{F}_{3\times 3} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \right) \cdot \left(\mathcal{F}^{-1}_{6\times 6} \begin{pmatrix} 0 \\ 0 \\ 3.5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 3.5 \\ 0 \\ 0 \\ 0 \\ 3.5 \end{pmatrix}$$

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Identification Example: One Fourier Coefficient

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- We only utilize 4 samples
- Computed Efficiently using 2 FFTs
- Reconstruct frequency index via Chinese Remainder Theorem
- Two estimates of nonzero Fourier coefficient

SAVED TWO SAMPLES!

Design Decision #3: Coefficient Estimation

Frequency Isolated in a Convolution

$$f_j(x) := \mathrm{Conv}[\mathbb{e}^{-\mathbb{i} n_j a x} g(x),\ \mathbb{e}^{\mathbb{i} m_{l_j} x} f(d_{l_j} x)](x) = C_j' \cdot \mathbb{e}^{x \cdot \omega_j' \cdot \mathbb{i}} + \epsilon(x)$$

1. Sometimes the procedure for identifying ω_j' automatically provides estimates of C_j' ...
2. If not, we can compute $C_j' \approx \mathbb{e}^{-x \cdot \omega_j' \cdot \mathbb{i}} f_j(x)$ if $\epsilon(x)$ small
3. Approximate C_j' via (Monte Carlo) integration techniques, e.g.,

$$C_j' \approx \int_0^{2\pi} \mathbb{e}^{-x \cdot \omega_j' \cdot \mathbb{i}}\ f_j(x)\, dx \approx \frac{1}{K} \sum_{h=1}^{K} \mathbb{e}^{-x_h \cdot \omega_j' \cdot \mathbb{i}}\ f_j(x_h)$$

What have we got so far?

Approximate $\left\{(\omega_j, C_j) \,\middle|\, 1 \leq j \leq k\right\}$ by sampling

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathbb{e}^{x \cdot \omega_j \cdot \mathbb{i}}, \; \Omega = \{\omega_1, \ldots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

1. We can isolate (a proxy for) each $\omega_j \in \Omega$, in some

$$f_j(x) = \mathrm{Conv}[\mathbb{e}^{-\mathbb{i}nax} g(x), \; \mathbb{e}^{\mathbb{i}m_l x} f(d_l x)](x)$$

   for some n, m_l, d_l triple with high probability (w.h.p.).
2. We can identify ω_j by, e.g., doing a binary search on $\hat{f}_j$
3. We can get a good estimate of C_j from $f_j(x)$ once we know ω_j

We have a lot of estimates, $\left\{(\tilde{\omega}_j, \tilde{C}_j) \,\middle|\, 1 \leq j \leq c_1 k \log^{c_2} N\right\}$, which contain the true Fourier frequency/coefficient pairs.

How do we discard the junk?

What have we got so far?

Approximate $\{(\omega_j, C_j) \mid 1 \leq j \leq k\}$ by sampling

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathrm{e}^{x \cdot \omega_j \cdot \mathrm{i}}, \; \Omega = \{\omega_1, \ldots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

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$$f_j(x) = \mathrm{Conv}[\mathrm{e}^{-\mathrm{i} nax} g(x), \; \mathrm{e}^{\mathrm{i} m_l x} f(d_l x)](x)$$

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We have a lot of estimates, $\left\{(\tilde{\omega}_j, \tilde{C}_j) \mid 1 \leq j \leq c_1 k \log^{c_2} N\right\}$, which contain the true Fourier frequency/coefficient pairs.

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How do we discard the junk?

What have we got so far?

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1. We can isolate (a proxy for) each $\omega_j \in \Omega$, in some

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How do we discard the junk?

Design Decision #4: Iteration?

Approximate $\left\{(\omega_j, C_j) \;\middle|\; 1 \leq j \leq k\right\}$ by sampling

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathrm{e}^{x \cdot \omega_j \cdot \mathrm{i}}, \; \Omega = \{\omega_1, \ldots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

- Analyzing probability of isolation is akin to considering tossing balls (frequencies of f) into bins (pass regions of modulated filter)

No Iteration: Identification and Estimation Once

Approximate $\{(\omega_j, C_j) \mid 1 \leq j \leq k\}$ by sampling

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathbb{e}^{x \cdot \omega_j \cdot \mathbb{i}}, \ \Omega = \{\omega_1, \ldots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

1. Tossing the balls (frequencies) into $O(k)$ bins (pass regions) about $T = O(\log N)$-times guarantees that each ball lands in a bin "by itself" on the majority of tosses, w.h.p.
   - Translation: We should identify dominant frequency of

     $$\mathrm{Conv}[\mathbb{e}^{-\mathbb{i}nax} g(x), \ \mathbb{e}^{\mathbb{i}m_l x} f(d_l x)](x)$$

     for $O(\log N)$ random (m_l, d_l)-pairs, $\forall n \in O([-k, k])$.
2. Will identify each $\omega_j \in \Omega$ for $> T/2$ (m_l, d_l)-pairs w.h.p.
3. SO,...we can take medians of real/imaginary parts of C_j estimates for each frequency identified by $> T/2$ (m_l, d_l)-pairs as our final Fourier coefficient estimate for that frequency, and do fine

No Iteration: Identification and Estimation Once

Approximate $\left\{(\omega_j, C_j) \;\middle|\; 1 \leq j \leq k\right\}$ by sampling

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathbb{e}^{x \cdot \omega_j \cdot \mathbb{i}}, \; \Omega = \{\omega_1, \ldots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

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No Iteration: Identification and Estimation Once

Approximate $\{(\omega_j, C_j) \mid 1 \leq j \leq k\}$ by sampling

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathrm{e}^{x \cdot \omega_j \cdot \mathrm{i}}, \ \Omega = \{\omega_1, \ldots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

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Several rounds of Identification and Estimation

Approximate $\left\{(\omega_j, C_j) \mid 1 \leq j \leq k\right\}$ by sampling

$$f(x) \approx \sum_{j=1}^{k} C_j \cdot \mathbb{e}^{x \cdot \omega_j \cdot \mathbb{i}}, \ \Omega = \{\omega_1, \ldots, \omega_k\} \subset \left(-\frac{N}{2}, \frac{N}{2}\right] \bigcap \mathbb{Z}$$

1. Tossing the balls (frequencies) into $O(k)$ bins (pass regions) about $O(T)$-times guarantees that each ball lands in a bin "by itself" at least once with probability $1 - 2^{-T}$
   - Idea: We should identify dominant frequency of
     $$\mathrm{Conv}[\mathbb{e}^{-\mathbb{i}nax} g(x), \ \mathbb{e}^{\mathbb{i}m_l x} f(d_l x)](x)$$
     for $O(1)$ random (m_l, d_l)-pairs, $\forall n \in O([-k, k])$.
   - We can expect to correctly identify a constant fraction of $\omega_1, \ldots, \omega_k$
2. Accurately estimating the Fourier coefficients of the identified frequencies is comparatively easy (no binary search required)
3. As long as we estimate the Fourier coefficients of the energetic frequencies "well enough", we've made progress

Several rounds of Identification and Estimation

Approximate $\left\{(\omega_j, C_j) \,\middle|\, 1 \leq j \leq k\right\}$ by sampling

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Round 2

1. If we made progress the first time, so we should do it again ...

Implicitly Create a “New Signal"

$$f_2(x) := f(x) - \sum_{j=1}^{O(k)} \tilde{C}_j \cdot \mathbb{e}^{x \cdot \tilde{\omega}_j \cdot \mathbb{i}} \approx \sum_{j=1}^{k/4} C'_j \cdot \mathbb{e}^{x \cdot \omega'_j \cdot \mathbb{i}},$$

where $(\tilde{\omega}_j, \tilde{C}_j)$ where obtained from the last round

2. Sparsity is effectively reduced. Repeat. . .

Round j

1. Tossing the remaining $k/4^j$ balls (frequencies) into $O(k/4^j)$ bins (pass regions) about $O(j)$-times guarantees that each remaining ball lands in a bin "by itself" at least once with probability $1 - 2^{-j}$
   - We should identify dominant frequencies of
   $$\mathrm{Conv}[\mathrm{e}^{-\mathrm{i}nax} g(x),\ \mathrm{e}^{\mathrm{i}m_l x} f(d_l x)](x)$$
   for $O(j)$ random (m_l, d_l)-pairs, $\forall n \in O([-k/4^j, k/4^j])$.
   - We identify a constant fraction of remaining frequencies, $\omega'_1, \ldots, \omega'_{k/4^j}$, with higher probability
2. Estimating Fourier coefficients of identified frequencies can be done more accurately (e.g., w/ relative error $O(2^{-j})$)

⋮

3. We eventually find all of $\omega_1, \ldots, \omega_k$ with high probability after $O(\log k)$-rounds. Samples/runtime will be dominated by first round IF. . . .

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1. Tossing the remaining $k/4^j$ balls (frequencies) into $O(k/4^j)$ bins (pass regions) about $O(j)$-times guarantees that each remaining ball lands in a bin "by itself" at least once with probability $1 - 2^{-j}$
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   - We should identify dominant frequencies of
   $$\mathrm{Conv}[\mathbb{e}^{-\mathbb{i} nax} g(x),\ \mathbb{e}^{\mathbb{i} m_l x} f(d_l x)](x)$$
   for $O(j)$ random (m_l, d_l)-pairs, $\forall n \in O([-k/4^j, k/4^j])$.
   - We identify a constant fraction of remaining frequencies, $\omega'_1, \ldots, \omega'_{k/4^j}$, with higher probability
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⋮

3. We eventually find all of $\omega_1, \ldots, \omega_k$ with high probability after $O(\log k)$-rounds. Samples/runtime will be dominated by first round IF. . . .

We Can Quickly Sample From Residual Signal

The Residual Signal We Need to Sample

$$f_j(x) := f(x) - \sum_{h=1}^{O(k)} \tilde{C}_h \cdot \mathbb{e}^{x \cdot \tilde{\omega}_h \cdot \mathbb{i}} \approx \sum_{h=1}^{k/4^j} C'_h \cdot \mathbb{e}^{x \cdot \omega'_h \cdot \mathbb{i}},$$

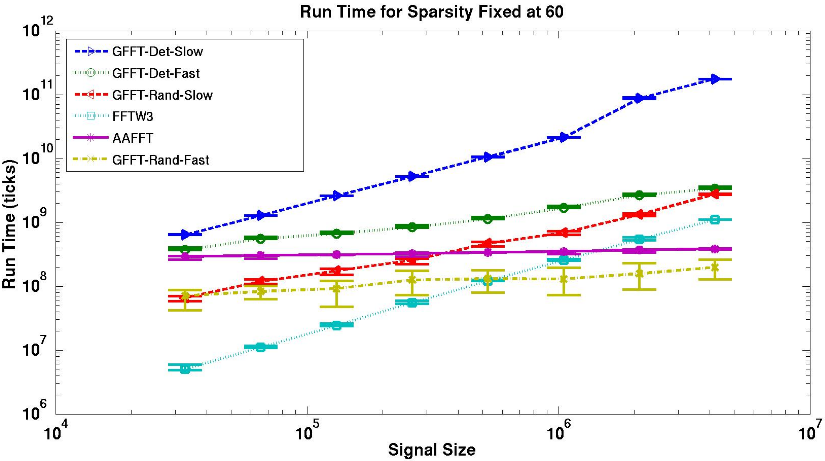
where $(\tilde{\omega}_h, \tilde{C}_h)$ where obtained from the previous rounds

- Subtracting Fourier terms from previous rounds, $(\tilde{\omega}_h, \tilde{C}_h)$, from each "frequency bin" they fall into
  - We know what filter's pass region each $\tilde{\omega}_h$ will fall into (e.g., call it n_h). Subtract $\tilde{C}_h$ from the Fourier transform of

    $$\mathrm{Conv}[\mathbb{e}^{-\mathbb{i} n_h a x} g(x),\ \mathbb{e}^{\mathbb{i} m_l x} f(d_l x)](x)$$

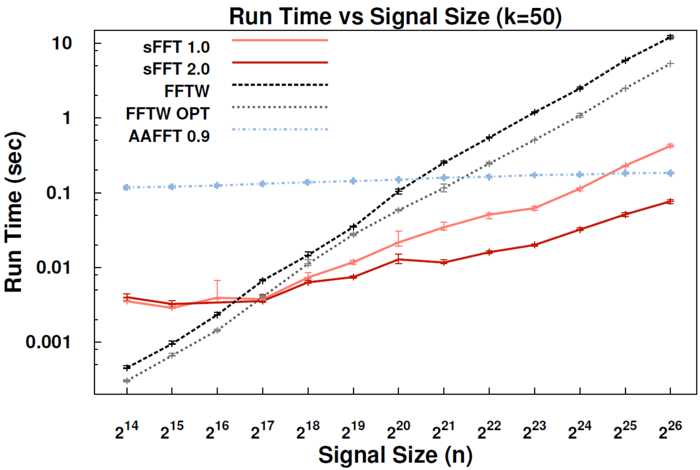
    for each (m_l, d_l)-pair during subsequent rounds.
- Or, we can use nonequispaced FFT ideas (several grids on arithmetic progressions, frequencies nonequispaced).

Publicly Available Codes: FFTW, AAFFT, and GFFT



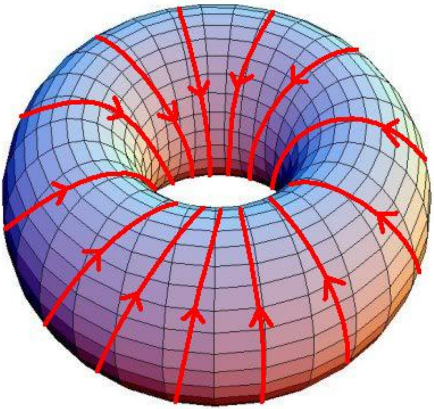
- FFTW: `http://www.fftw.org`
- AAFFT, GFFT: `http://sourceforge.net/projects/gopherfft/`

Publicly Available Codes: SFT 1.0 and 2.0



- http://groups.csail.mit.edu/netmit/sFFT/code.html

Extending to Many Dimensions



- Sample $f^{\text{new}}(x) = f\left(x\frac{\tilde{N}}{P_1}, \dots, x\frac{\tilde{N}}{P_D}\right)$, with $\tilde{N} = \prod_{d=1}^{D} P_d > N^D$
- Works because $\mathbb{Z}_{\tilde{N}}$ is isomorphic to $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_D}$.

Questions?

Thank You!

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