Any of the following exercises are fair game for appearing in a slightly altered state on an exam. I suggest that you write up your solutions **neatly** in your own handwriting, or .tex them up, so that you can easily use them for studying later. All solutions should be your own! Feel free to discuss solutions with others, but don't write anything down that you don't understand (and write it all yourself!). **Only a subset of the problems will be graded, so do them all to the best of your ability, and be aware: even if there is only one problem that you don't do well, it might be one that's graded! So, try to do them all well!**

PROBLEMS ASSIGNED ABOUT DISCRETE FOURIER TRANSFORMS AND CONVOLUTIONS

1. Show that $\mathbb{e}^{x+\mathbb{i}2\pi\theta} = \mathbb{e}^x$ holds for all $\theta \in \mathbb{Z}$ and $x \in \mathbb{C}$. As a consequence, conclude, for example, that $\mathbb{e}^{\mathbb{i}\pi\theta} = \mathbb{e}^{-\mathbb{i}\pi\theta} = (-1)^\theta$ for all $\theta \in \mathbb{Z}$.
2. Show that $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=0}^{N-1} x_j \overline{y_j}$ is an inner product for $\mathbb{C}^N$.
3. Prove that the DFT matrix, $F \in \mathbb{C}^{N\times N}$, is unitary.
4. Show that $\|\hat{\mathbf{v}}\|_2 = \|\mathbf{v}\|_2$ holds for all $\mathbf{v} \in \mathbb{C}^N$. Here $\|\mathbf{v}\|_2 := \sqrt{\sum_j |v_j|^2}$ is $\mathbf{v}$'s Euclidean norm.
5. Let $\mathbf{v}, \mathbf{u} \in \mathbb{C}^N$. Show that $\mathbf{u} \star \mathbf{v} = \mathbf{v} \star \mathbf{u}$ holds (i.e., order doesn't matter).
6. Let $a, b, c \in [N]$ be such that a is invertible modulo N.[1] Furthermore, suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$ satisfy
$$v_j = \mathbb{e}^{\frac{2\pi \mathbb{i} cj}{N}} u_{aj+b \mod N}$$
for all $j \in [N]$. Calculate $\hat{\mathbf{v}}$ in terms of $\hat{\mathbf{u}}$. How does a affect $\hat{\mathbf{v}}$ when $c = b = 0$? How does b affect $\hat{\mathbf{v}}$ when $a = 1$ and $c = 0$? How does c affect $\hat{\mathbf{v}}$ when $a = 1$ and $b = 0$?
7. Let $\omega \in \mathbb{Z}$. Show that
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{e}^{\mathbb{i}\omega x}\, dx \;=\; \begin{cases} 1 \text{ if } \omega = 0 \\ 0 \text{ if } \omega \neq 0 \end{cases}.$$
8. Let $p : [-\pi, \pi] \to \mathbb{C}$ be a degree N trigonometric polynomial given by $p(x) = \sum_{\omega=-N}^{N} \hat{p}_\omega \mathbb{e}^{\mathbb{i}\omega x}$. Show that
$$\|p\|_2 := \sqrt{\int_{-\pi}^{\pi} |p(x)|^2\, dx} = \sqrt{\sum_{\omega=-N}^{N} |\hat{p}_\omega|^2} =: \|(\hat{p}_{-N}, \ldots, \hat{p}_0, \ldots, \hat{p}_N)\|_2 = \|\ \mathcal{F}[p]\ \|_2 .$$
9. Let $g : [0,1] \to \mathbb{R}$ be a twice continuously differentiable and periodic function. We will prove a little later that any such g will have a *Fourier series expansion* of the form
$$g(x) = \sum_{\omega\in\mathbb{Z}} c_\omega \mathbb{e}^{2\pi\mathbb{i}\omega x} \quad \forall x \in [0,1], \tag{1}$$
where the *Fourier series coefficients* $c_\omega \in \mathbb{C}$ satisfy (i) $c_\omega = \overline{c_{-\omega}}$ for all $\omega \in \mathbb{Z}$, and (ii) $\sum_{\omega\in\mathbb{Z}} |c_\omega| < \infty$. Let $\mathbf{u} \in \mathbb{R}^N$ be a vector whose entries are given by $u_j = g(j/N)$ for all $j \in [N]$. Show that the vector $F\mathbf{u}$ has entries
$$(F\mathbf{u})_j = \sqrt{N} \sum_{\omega \equiv j \mod N} c_\omega.$$

[1] A value $a \in [N]$ is invertible modulo N if there exists an $h \in [N]$ such that $(a\ h) \equiv 1 \mod N$. Any $a \in [N]$ that is relatively prime to N will be invertible modulo N by the Fermat-Euler Theorem.