Do any 7 of the following 10 exercises of your choice. Write up your solutions neatly in your own handwriting! Many problems are from the book, which is available online here: `http://www-elec.inaoep.mx/~rogerio/FourierAnalysisUno.pdf`. Students who are interested in understanding the material more completely are encouraged to solve problems 1, 2, 3, 4, 9, and 10 as part of their homework.

1. Do problem 1 on page 61 of Folland. For part (c) you should look up and apply the Dominated Convergence Theorem in order to make your solution rigorous. **This problem shows that a function's Fourier series may not converge uniformly to the function on $[-\pi, \pi]$ even if it is 2π periodic, piecewise smooth, and has only one discontinuity!** The upshot: continuity is both a necessary and sufficient condition for the uniform convergence of an arbitrary 2π periodic and piecewise smooth function's Fourier series to the function everywhere on $[-\pi, \pi]$ (see Theorem 2.5 on page 41 of Folland for sufficiency).

2. Recall the we defined the class $\mathcal{C}^k$ of functions $f : [-\pi, \pi] \to \mathbb{C}$ to be those with a Riemann integrable k^{th} derivative on $[-\pi, \pi]$. We are also assuming periodicity so that all lesser derivatives of f, $f^{(l)}$, are continuous with $f^{(l)}(\pi) = f^{(l)}(-\pi)$ for all $l = 0, \ldots, k-1$. For any $f \in \mathcal{C}^k$ we then showed that:

   (a) **THM 1/28 (A):** $c_n = c_n^{(k)}/(in)^k$ for all $n \neq 0$ (where $c_n^{(l)}$ is the n^{th} Fourier series coefficient of $f^{(l)}$, with $c_n = c_n^{(0)}$).

   (b) **THM 1/26:** If $f \in \mathcal{C}^k$ with $k \geq 2$, then $f(\theta) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \mathbb{e}^{\mathbb{i}n\theta}$ for all $\theta \in [-\pi, \pi]$ (pointwise). If fact, we proved something a bit more general than this...but this will do here.

   Use these two facts to prove that, for every $f \in \mathcal{C}^k$ with $k \geq 2$, there exists a constant a $B \in \mathbb{R}^+$ (which only depends on $f^{(k)}$) such that

   $$\left| f(\theta) - \sum_{n=-N}^{N} c_n \mathbb{e}^{\mathbb{i}n\theta} \right| \leq \frac{B}{N^{k-1}} \tag{1}$$

   holds for all $N \in \mathbb{Z}^+$ and $\theta \in [-\pi, \pi]$. This proves that convergence of the Fourier series is both uniform as soon as f is smooth enough, and also generally faster the smoother f is.

3. The theorem you proved for #2 is useful in applications. Let's try to understand it a little better with an example: Consider the function $f(x) = \cos(100x)$. Note that $f \in \mathcal{C}^k$ for any $k \in \mathbb{Z}^+$ you like.

   (a) Look back at your proof of Theorem 2 and figure out how large B is for $f(x) = \cos(100x)$ when $k = 13$. How does this B vary with k in general?

   (b) How large does the error bound in equation 1 tell you have to take N before you can be sure that your error will always be less than .001 for all $\omega \in [-\pi, \pi]$ when using $k = 13$ for $f(x) = \cos(100x)$?

   (c) How large will your actual error be once you pick any $N \geq 100$ for $f(x) = \cos(100x)$?

   (d) How large does your error bound in equation 1 tell you have to take N before you can be sure that your error will always be less than 10^{-16} for all $\omega \in [-\pi, \pi]$ as you let $k \to \infty$ for $f(x) = \cos(100x)$?

4. Suppose you know that a function $f : [-\pi, \pi] \to \mathbb{C}$ you want to learn about is composed of exactly one frequency component. That is, that $f(x) = A\mathbb{e}^{\mathbb{i}\omega x}$ for unknown parameters $A \in \mathbb{C}$ and $\omega \in \mathbb{Z} \cap [-127627, 127627]$. Use **THM 2/4** from class (see below) together with the Chinese Remainder Theorem in order to show that you can learn both A and ω by sampling f at just 51 different points $\in [-\pi, \pi]$.

(a) **THM 2/4:** Let $\tilde{c}_n := \frac{(-1)^n}{N} \sum_{k=0}^{N-1} f\left(-\pi + k \cdot \frac{2\pi}{N}\right) \mathrm{e}^{\frac{-2\pi i n k}{N}}$. Then,

$$\tilde{c}_n = \sum_{q=-\infty}^{\infty} c_{n+Nq} = \sum_{m \equiv n \bmod N} c_m,$$

where c_n is the n^{th} Fourier series coefficient of $f : [-\pi, \pi] \to \mathbb{C}$.

HINT: You will want to use 6 sets of equally spaced samples in $[-\pi, \pi]$, each associated with a different prime number.

5. Do problems 6 and 7 on page 68 of Folland.
6. Do problems 8 and 9 on page 68 of Folland.
7. Do problems 1 and 2 on page 71 of Folland.
8. Do problem 3 on page 71 of Folland.
9. Do problem 4 on page 71 of Folland.
10. Do problems 6 and 7 on page 71 of Folland.