Exercises:

§19, 20

1. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $a_n > 0$ for all $n \in \mathbb{N}$. Define a map $h\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$h((x_n)_{n\in\mathbb{N}}) = (a_n x_n + b_n)_{n\in\mathbb{N}}.$$

   (a) Show that h is a bijection.

   (b) Show that if $\mathbb{R}^{\mathbb{N}}$ is given the product topology, then h is a homeomorphism.

   (c) Prove whether or not h is a homeomorphism when $\mathbb{R}^{\mathbb{N}}$ is given the box topology.

2. For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, define

$$d_1(\mathbf{x}, \mathbf{y}) := \sum_{j=1}^{n} |x_j - y_j|.$$

   (a) Show that d_1 is a metric on $\mathbb{R}^n$.

   (b) Show that the topology induced by d_1 equals the product topology on $\mathbb{R}^n$.

   (c) For $n = 2$ and $\mathbf{0} = (0, 0) \in \mathbb{R}^2$, draw a picture of $B_{d_1}(\mathbf{0}, 1)$.

3. Let X be a metric space with metric d. For $x \in X$ and $\epsilon > 0$, show that $\{y \in X \mid d(x, y) \le \epsilon\}$ is a closed set.

4. Let X be a metric space with metric d. Show that $d\colon X \times X \to \mathbb{R}$ is continuous.

5. For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$ define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &:= (x_1 + y_1, \ldots, x_n + y_n),\\
c\mathbf{x} &:= (cx_1, \ldots, cx_n),\\
\mathbf{x} \cdot \mathbf{y} &:= x_1 y_1 + \cdots + x_n y_n,\\
\|\mathbf{x}\| &:= (x_1^2 + \cdots + x_n^2)^{1/2}.
\end{aligned}$$

   (a) For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, prove the following formulas

$$\begin{aligned}
\|\mathbf{x}\|^2 &= \mathbf{x} \cdot \mathbf{x}\\
(a\mathbf{x}) \cdot (b\mathbf{y}) &= (ab)(\mathbf{x} \cdot \mathbf{y})\\
\mathbf{x} \cdot \mathbf{y} &= \mathbf{y} \cdot \mathbf{x}\\
\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}
\end{aligned}$$

   (b) Show that $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|$. [**Hint:** for $\mathbf{x}, \mathbf{y} \neq 0$ let $a = \frac{1}{\|\mathbf{x}\|}$ and $b = \frac{1}{\|\mathbf{y}\|}$ and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\|^2 \ge 0$.]

   (c) Show that $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

   (d) Prove that the euclidean metric $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|$ is indeed a metric.

6*. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \le p < \infty$, define

$$\|\mathbf{x}\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p},$$

and for $p = \infty$ define

$$\|\mathbf{x}\|_\infty := \max\{|x_1|, \ldots, |x_n|\}.$$

In this exercise you will show $d_p(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_p$ defines a metric for each $1 \le p \le \infty$. Observe that $p = 1, 2, \infty$ yield the metric from Exercise 2, the euclidean metric, and the square metric, respectively.

(a) For $1 < p < \infty$, show that if $q > 0$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$ then $1 < q < \infty$. We call q the **conjugate exponent** to p.

(b) For $a, b \geq 0$ and $0 < \lambda < 1$, show that $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$.

(c) Prove **Hölder's Inequality**: for $1 < p < \infty$ with conjugate exponent q and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ show that

$$|x_1 y_1| + \cdots + |x_n y_n| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

(d) Prove **Minkowski's Inequality**: for $1 < p < \infty$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ show that

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$

[**Hint:** use $|x_j + y_j|^p \leq (|x_j| + |y_j|)|x_j + y_j|^{p-1}$.]

(e) Show that d_p is a metric for $1 < p < \infty$.

(f) Show that the topology induced by d_p equals the product topology on $\mathbb{R}^n$ for $1 < p < \infty$, where $\mathbb{R}$ has the standard topology. [**Hint:** show that $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_1$.]

* Challenge Problem!