Exercises:

§18, 19

1. Let A, B, C, D be topological spaces and suppose $f\colon A \to B$ and $g\colon C \to D$ are continuous functions. Define a function $f \times g\colon A \times C \to B \times D$ by

$$(f \times g)(a, c) = ((f(a), g(c)).$$

   Show that $f \times g$ is continuous when $A \times C$ and $B \times D$ are given the product topologies.

2. Let $\mathbb{R}$ and $\mathbb{R}^2$ have their standard topologies.
   (a) Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$ is continuous.
   (b) For each $n \in \mathbb{N}$, show that $p\colon \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^n$ is continuous.

3. Let X be a topological space and let Y be a set with an order relation $<$ and the order topology. Suppose $f, g\colon X \to Y$ are continuous.
   (a) Show that the set $\{x \in X \mid f(x) \leq g(x)\}$ is closed in X.
   (b) Show that the function $h\colon X \to Y$ defined by $h(x) := \min\{f(x), g(x)\}$ is continuous. [**Hint:** using the pasting lemma.]

4. Let $\mathbb{R}$ have the standard topology. Consider

$$C = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid x_n \neq 0 \text{ for only finitely many } n \in \mathbb{N}\}.$$

   That is, C is the set of sequences that are eventually equal to zero.
   (a) Determine $\overline{C}$ when $\mathbb{R}^{\mathbb{N}}$ has the box topology.
   (b) Determine $\overline{C}$ when $\mathbb{R}^{\mathbb{N}}$ has the product topology.

5. Let $\mathbb{R}$ have the standard topology and consider the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

   and

$$g(x) = \begin{cases} \frac{1}{m} & x \in \mathbb{Q} \text{ with } x = \frac{n}{m} \text{ for } n \in \mathbb{Z} \text{ and } m \in \mathbb{N} \text{ sharing no common factors} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

   (a) Show that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.
   (b) Show that f is **not** continuous at any $x \in \mathbb{R}$.
   (c) Show that g is **not** continuous at any $x \in \mathbb{Q}$.
   (d) Show that g is continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$.