Exercises:

§17, 18

1. Prove each of the following topological spaces is Hausdorff.
   (a) A set X with an order relation $<$ and the order topology.
   (b) A product $X \times Y$ with the product topology where X and Y are Hausdorff spaces.
   (c) A subspace $Y \subset X$ with the subspace topology where X is a Hausdorff space.

2. Let X be a topological space. Show that X is Hausdorff if and only if the **diagonal**
$$\Delta := \{(x, x) \mid x \in X\}$$
is a closed subset of $X \times X$ with the product topology.

3. Consider the collection $\mathcal{T} = \{U \subset \mathbb{R} \mid \mathbb{R} \setminus U \text{ is finite}\} \cup \{\emptyset\}$.
   (a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$. We call this the **finite complement topology**.
   (b) Show that the finite complement topology is T_1: given distinct points $x, y \in \mathbb{R}$ there exists open sets U and V with $x \in U \not\ni y$ and $x \notin V \ni y$.
   (c) Show that the finite complement topology is not Hausdorff.
   (d) Find all the points that the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to in the finite complement topology.

4. Let X be a set with two topologies $\mathcal{T}$ and $\mathcal{T}'$ and let $i \colon X \to X$ be the identity function: $i(x) = x$ for all $x \in X$. Equip the domain copy of X with the topology $\mathcal{T}$ and the range copy of X with the topology $\mathcal{T}'$.
   (a) Show that i is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}'$.
   (b) Show that i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.

5. Consider the functions $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x, y) = x + y \qquad \text{and} \qquad g(x, y) = x - y.$$
   (a) Show that if $\mathbb{R}$ and $\mathbb{R}^2$ are given the standard topologies, then f and g are continuous.
   (b) Suppose $\mathbb{R}$ is given the lower limit topology and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is given the corresponding product topology. Determine and prove the continuity or discontinuity of f and g.

6*. In this exercise you will establish a homeomorphism between the following two subspaces of $\mathbb{R}^2$:
$$X := \mathbb{R}^2 \setminus \{(0, 0)\} \qquad \text{and} \qquad Y := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1\}.$$
Throughout, $\mathbb{R}^2$ will have the standard topology and X and Y will have their subspace topologies.
   (a) Define a function $\|\cdot\| \colon \mathbb{R}^2 \to [0, +\infty)$ by $\|(x, y)\| = (x^2 + y^2)^{1/2}$. Show that this function is continuous when $[0, +\infty) \subset \mathbb{R}$ is given the subspace topology. [**Hint:** think geometrically.]
   (b) Show that $X = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| > 0\}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| > 1\}$.
   (c) Show that $f \colon X \to \mathbb{R}^2$ defined by $f(x, y) = \frac{1}{\|(x,y)\|}(x, y)$ is continuous.
   (d) Find continuous functions $g \colon X \to Y$ and $h \colon Y \to X$ satisfying $g \circ h(x, y) = (x, y)$ and $h \circ g(x, y) = (x, y)$, and deduce that X and Y are homeomorphic.

* Challenge Problem!