Exercises: (§14, §15, §16, and §17)

1. Let X be an ordered set (with at least two elements) equipped with the order topology. For a subspace $Y \subset X$, show that the collection $\mathcal{S}$ consisting of sets of the form $Y \cap (-\infty, a)$ or $Y \cap (a, +\infty)$ for $a \in X$ form a subbasis for the subspace topology on Y.

2. Let X and Y be topological spaces. A function $f\colon X \to Y$ is called an **open map** if for every open subset $U \subset X$ one has that its image $f(U)$ is open in Y.

   (a) Equip $X \times Y$ with the product topology. Show that the coordinate projections $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are open maps.

   (b) Let $\mathcal{B}$ be a basis for the topology on X and suppose $f(B)$ is open for all $B \in \mathcal{B}$. Show that f is an open map.

   (c) Show that the previous part does not hold for subbases. [**Hint:** consider the function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 1$ and $f(x) = |x|$ if $x \neq 0$ where $\mathbb{R}$ has the standard topology.]

3. Show that if Y is a topological subspace of X, and $A \subset Y$, then the topology A inherits as a subspace of Y is the same as the topology that it inherits as a subspace of X.

4. Equip $\mathbb{R}$ with the standard topology.

   (a) Show that the subspace topology on $\{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is the discrete topology.

   (b) Show that the subspace topology on $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is **not** the discrete topology.

5. Let $\mathcal{C}$ be a collection of subsets of X. Assume that $\emptyset, X \in \mathcal{C}$ and that finite unions and arbitrary intersections of sets in $\mathcal{C}$ are in $\mathcal{C}$. Show that the collection $\mathcal{T} := \{X \setminus C \mid C \in \mathcal{C}\}$ is a topology on X and that the collection of closed sets in this topology is $\mathcal{C}$.