Exercises: (§12 and §13)

1. Let X be a topological space and let $A \subset X$ be a subset. Suppose that for all $x \in A$, there exists an open set U satisfying $x \in U \subset A$. Show that A is open.

2. Equip $\mathbb{R}$ with the standard topology. Show that a set $U \subset \mathbb{R}$ is open if and only if for all $x \in U$ there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$.

3. Show that the collection $\mathcal{B} = \{(a, b) \colon a, b \in \mathbb{Q}\}$ is a basis for the standard topology on $\mathbb{R}$. Conclude that standard topology on $\mathbb{R}$ therefore has a countable basis.

4. Let X be a space.
   (a) Let $\{\mathcal{T}_i \mid i \in I\}$ be a non-empty collection topologies on X (indexed by some set I). Show that $\bigcap_{i \in I} \mathcal{T}_i$ is a topology on X.
   (b) Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Show that $\mathcal{T}$ is the intersection of all topologies on X that contain $\mathcal{B}$.
   (c) Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on a space X. Suppose $\mathcal{T}'$ is another topology on X that contains $\mathcal{S}$. Show that $\mathcal{T}$ is coarser than $\mathcal{T}'$.
   (d) Let $\mathcal{S}$ and $\mathcal{T}$ be as in the previous part. Show that $\mathcal{T}$ is the intersection of all topologies on X that contain $\mathcal{S}$.