<u>Exercises:</u> (§9, §10, and §11)

1. Let $f\colon A \to B$ be a function.
   (a) Use the axiom of choice to show that if f is surjective, then there exists $g\colon B \to A$ with $f \circ g(b) = b$ for all $b \in B$.
   (b) Without using the axiom of choice show that if f is injective, then there exists $h\colon B \to A$ with $h \circ f(a) = a$ for all $a \in A$.
2. Show that the well-ordering theorem implies the axiom of choice.
3. Let S_Ω be the minimal uncountable well-ordered set from §10.
   (a) Show that S_Ω has no largest element.
   (b) Show that for every $x \in S_\Omega$, the subset $\{y \in S_\Omega \mid x < y\}$ is uncountable.
   (c) Consider the subset
   $$X := \{x \in S_\Omega \mid \text{ the open interval } (a, x) \neq \emptyset \text{ for all } a < x\}.$$
   Show that X is uncountable. [**Hint:** proceed by contradiction and use the fact that for any $y \in S_\Omega$ there exists $z \in S_\Omega$ with the open interval $(y, z) = \emptyset$.]
4. In this exercise you will use Zorn's lemma to prove the following fact from linear algebra: every vector space V has a basis. For a subset $A \subset V$, recall: the **span** of A is the set of all finite linear combinations of vectors in A; A is said to be **independent** if the only way to write the zero vector as a linear combination of elements in A is via the trivial linear combination with all zero scalar coefficients; and A is said to be a **basis** for V if it is independent and its span is all of V.
   (a) Suppose $A \subset V$ is independent. Show that if v is not in the span of A, then $A \cup \{v\}$ is independent.
   (b) Show that the collection of independent subsets of V, ordered by inclusion, has a maximal element.
   (c) Show that V has a basis.