Exercises:

§23, 24, 26

1. Let X be a topological space and let $\{Y_j \mid j \in J\}$ be an indexed family of connected subspaces of X. Suppose there exists a connected subspace $Y \subset X$ satisfying $Y \cap Y_j \neq \emptyset$ for all $j \in J$. Show that $Y \cup \bigcup_{j \in J} Y_j$ is connected.

2. Recall that $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

   (a) Show that S^1 is connected.

   (b) Show that $a(x, y) := (-x, -y)$ defines a homeomorphism $a\colon S^1 \to S^1$.

   (c) Show that if $f\colon S^1 \to \mathbb{R}$ is continuous, then there exists $(x, y) \in S^1$ satisfying $f(x, y) = f(-x, -y)$.

3. Let $U \subset \mathbb{R}^n$ be open and connected. Show that U is path connected.

   [**Hint:** for $\mathbf{x}_0 \in U$ show that the set of points $\mathbf{x} \in U$ that are connected to $\mathbf{x}_0$ by a path in U is clopen.]

4. Equip $\mathbb{R}$ with the finite complement topology. Show that every subset is compact.

5. Let X be a Hausdorff space. If $A, B \subset X$ are compact with $A \cap B = \emptyset$, show that there are open sets $U \supset A$ and $V \supset B$ with $U \cap V = \emptyset$.

6. Let $p\colon X \to Y$ be a closed continuous surjective map.

   (a) For $U \subset X$ open, show that $p^{-1}(\{y\}) \subset U$ for $y \in Y$ implies there is a neighborhood V of y with $p^{-1}(V) \subset U$.

   (b) Show that if Y is compact and $p^{-1}(\{y\})$ is compact for each $y \in Y$, then X is compact.