Exercises:

§22

1. Let X be a topological space. For a subset $A \subset X$, a **retraction** of X onto A is a continuous map $r\colon X \to A$ satisfying $r(a) = a$ for all $a \in A$.
   (a) Let $p\colon X \to Y$ be a continuous map between topological spaces. Show that if there exists a continuous function $f\colon Y \to X$ so that $p(f(y)) = y$ for all $y \in Y$, then p is a quotient map.
   (b) Show that a retraction is a quotient map.

2. Consider the following subset of $\mathbb{R}^2$:
$$A := \{(x, y) \in \mathbb{R}^2 \mid \text{either } x \geq 0 \text{ or } y = 0 \text{ (or both)}\}.$$
   Define $q\colon A \to \mathbb{R}$ by $q(x, y) = x$. Show that q is a quotient map, but is neither open nor closed.

3. Let X and Y be topological spaces and let $p\colon X \to Y$ be a surjective map.
   (a) Show that a subset $A \subset X$ is saturated with respect to p if and only if $X \setminus A$ is saturated with respect to p.
   (b) Show that $p(U) \subset Y$ is open for all saturated open sets $U \subset X$ if and only if $p(A) \subset Y$ is closed for all saturated closed sets $A \subset X$.
   (c) Show that if p is an injective quotient map, then it is a homeomorphism.

4. Let $X := (0, 1] \cup [2, 3)$, $Y := (0, 2)$, and $Z := (0, 1] \cup (2, 3)$ and define maps $p\colon X \to Y$ and $q\colon X \to Z$ by
$$p(t) := \begin{cases} t & \text{if } 0 < t \leq 1 \\ t - 1 & \text{if } 2 \leq t < 3 \end{cases} \qquad \text{and} \qquad q(t) := \begin{cases} t & \text{if } t \neq 2 \\ 1 & \text{otherwise} \end{cases}.$$
   Equip X and Y with their subspace topologies from $\mathbb{R}$ and equip Z with the quotient topology induced by q.
   (a) Show that p is a quotient map.
   (b) Show that q is a quotient map.
   (c) Show that $f\colon Y \to Z$ defined by
$$f(t) := \begin{cases} t & \text{if } 0 < t \leq 1 \\ t + 1 & \text{if } 1 < t < 2 \end{cases}$$
   is a homeomorphism. [**Hint:** show $f \circ p = q$.]

5. Recall that for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, its norm is $\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$. Consider $X := \mathbf{R}^2 \setminus \{(0, 0)\}$ and $S^1 := \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\}$ equipped with their subspace topologies, where $\mathbb{R}^2$ has its standard topology.
   (a) Show that $p(\mathbf{x}) := \frac{1}{\|\mathbf{x}\|}\mathbf{x}$ defines a continuous map $p\colon X \to S^1$.
   (b) Show that p is a quotient map.
   (c) Define an equivalence relation $\sim$ on X so that the quotient space $X/\sim$ is homeomorphic to S^1. Give a geometric description of the equivalence classes.

6*. Consider
$$X := \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$$
$$S^2 := \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1\}.$$
   In this exercise you will show a quotient space of X is homeomorphic to S^2.

(a) Let $S^1 := \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\}$. Show that $f\colon X \setminus S^1 \to \mathbb{R}^2$ defined by

$$f(\mathbf{x}) := \frac{1}{1 - \|\mathbf{x}\|}\mathbf{x}$$

is a homeomorphism.

(b) Show that $g\colon S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ defined by

$$g(\mathbf{x}) := \frac{1}{1 - x_3}(x_1, x_2)$$

is a homeomorphism.

(c) Show that $p\colon X \to S^2$ defined by

$$p(\mathbf{x}) := \begin{cases} g^{-1} \circ f(\mathbf{x}) & \text{if } \mathbf{x} \in X \setminus S^1 \\ (0,0,1) & \text{otherwise} \end{cases}$$

is a quotient map.

(d) Define an equivalence relation on X by $\mathbf{x} \sim \mathbf{y}$ if and only if $p(\mathbf{x}) = p(\mathbf{y})$. Describe the quotient space $X/\sim$ and show that it is homeomorphic to S^2.

* Challenge Problem!