Exercises:

§20, 21

1. Let X be a metric space with metric d. Prove the **reverse triangle inequality**: for all $x, y, z \in X$
$$|d(x,y) - d(y,z)| \leq d(x,z).$$

2. Recall that the uniform metric on $\mathbb{R}^{\mathbb{N}}$ is defined as
$$\overline{\rho}(\mathbf{x},\mathbf{y}) = \sup_{n\in\mathbb{N}} \overline{d}(x_n, y_n),$$
where $\overline{d}(x_n, y_n) = \min\{|x_n - y_n|, 1\}$ is the standard bounded metric on $\mathbb{R}$.
   (a) Show that $\overline{\rho}$ is a metric.
   (b) Let $C \subset \mathbb{R}^{\mathbb{N}}$ be the subset from Exercise 4 on Homework 8. Determine $\overline{C}$ when $\mathbb{R}^{\mathbb{N}}$ has the topology induced by $\overline{\rho}$.
   (c) Let $h\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the function from Exercise 1 on Homework 9. Find necessary and sufficient conditions on the sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ which guarantee h is continuous when $\mathbb{R}^{\mathbb{N}}$ has the topology induced by $\overline{\rho}$.
   (d) For $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ and $\epsilon > 0$, show that
$$U := (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \cdots$$
   is **not** open with respect to the topology induced by $\overline{\rho}$.

3. Let X be a metric space with metric d. For fixed $x_0 \in X$, show that the function $f\colon X \to \mathbb{R}$ defined by $f(x) = d(x, x_0)$ is continuous.

4. For each $n \in \mathbb{N}$, define $f_n\colon \mathbb{R} \to \mathbb{R}$ by
$$f_n(x) = \frac{1}{1 + (x-n)^2}.$$
Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to the zero function pointwise but **not** uniformly.

5. Let X be a set and let Y be a metric space with metric d. Define a metric on Y^X by
$$\overline{\rho}((y_x)_{x\in X}, (z_x)_{x\in X}) := \sup_{x\in X} \overline{d}(y_x, z_x),$$
where $\overline{d}(y,z) = \min\{d(y,z), 1\}$ is the standard bounded metric corresponding to d. Let $f_n, f\colon X \to Y$ be functions, $n \in \mathbb{N}$, and define $\mathbf{f_n}, \mathbf{f} \in Y^X$ by $\mathbf{f_n} := (f_n(x))_{x\in X}$ and $\mathbf{f} := (f(x))_{x\in X}$. Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f if and only if the sequence $(\mathbf{f_n})_{n\in\mathbb{N}}$ converges to $\mathbf{f}$ when Y^X is given the topology induced by the metric $\overline{\rho}$.

6*. Let $\ell^2 \subset \mathbb{R}^{\mathbb{N}}$ be the set of sequences $(x_n)_{n\in\mathbb{N}}$ for which the series $\sum_{n=1}^{\infty} x_n^2$ converges. For $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \ell^2$ denote
$$\|\mathbf{x}\|_2 := \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}.$$
   (a) For $\mathbf{x} \in \ell^2$ and $c \in \mathbb{R}$, show that $c\mathbf{x} \in \ell^2$ with $\|c\mathbf{x}\|_2 = |c|\|\mathbf{x}\|_2$.
   (b) For $\mathbf{x}, \mathbf{y} \in \ell^2$, show that the series $\sum_{n=1}^{\infty} |x_n y_n|$ converges and is bounded by $\|\mathbf{x}\|_2\|\mathbf{y}\|_2$.
   (c) For $\mathbf{x}, \mathbf{y} \in \ell^2$, show that $\mathbf{x} + \mathbf{y} \in \ell^2$ with $\|\mathbf{x} + \mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$.
   (d) Show that $d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2$ defines a metric on ℓ^2.
   (e) Show that the topology induced by d_2 is finer than the uniform topology but coarser than the box topology on ℓ^2.

* Challenge Problem!