## Exercises: (§2 and §3)

1. Let $f\colon A \to B$ be a function.
   (a) For $A_0 \subset A$ and $B_0 \subset B$, show that $A_0 \subset f^{-1}(f(A_0))$ and $f(f^{-1}(B_0)) \subset B_0$.
   (b) Show that $f$ is injective if and only if $A_0 = f^{-1}(f(A_0))$ for all subsets $A_0 \subset A$.
   (c) Show that $f$ is surjective if and only if $f(f^{-1}(B_0)) = B_0$ for all subsets $B_0 \subset B$.

2. Let $f\colon A \to B$ be a function, $A_j \subset A$ for all $j \in \mathbb{Z}$, and $B_j \subset B$ for all $j \in \mathbb{Z}$. Prove the following:
   (a) $A' \subset A_0 \implies f(A') \subset f(A_0)$
   (b) $B' \subset B_0 \implies f^{-1}(B') \subset f^{-1}(B_0)$
   (c) $f(\cup_{j\in\mathbb{Z}} A_j) = \cup_{j\in\mathbb{Z}} f(A_j)$
   (d) $f^{-1}(\cup_{j\in\mathbb{Z}} B_j) = \cup_{j\in\mathbb{Z}} f^{-1}(B_j)$
   (e) $f(\cap_{j\in\mathbb{Z}} A_j) \subset \cap_{j\in\mathbb{Z}} f(A_j)$, where equality holds if $f$ is injective.
   (f) $f^{-1}(\cap_{j\in\mathbb{Z}} B_j) = \cap_{j\in\mathbb{Z}} f^{-1}(B_j)$ always

3. Let $C$ be a relation on a set $A$. For a subset $A_0 \subset A$, the **restriction** of $C$ to $A_0$ is the relation defined by the subset $D := C \cap (A_0 \times A_0)$.
   (a) For $a, b \in A$, show that $aDb$ if and only if $a, b \in A_0$ and $aCb$.
   (b) Show that if $C$ is an equivalence relation on $A$, then $D$ is an equivalence relation on $A_0$.
   (c) Show that if $C$ is an order relation on $A$, then $D$ is an order relation on $A_0$.
   (d) Show that if $C$ is a partial order relation on $A$, then $D$ is a partial order relation on $A_0$.

4. Let $f : A \to B$ be onto. Define a relation on $A$ by setting $a \sim a'$ if $f(a) = f(a')$. Show that $\sim$ is an equivalence relation. Furthermore, show that there is a bijection between the equivalence classes of $\sim$ and $B$.

5. We say two sets $A$ and $B$ have the same **cardinality** if there is a bijection of $A$ with $B$. In this exercise, you will prove the *Schröder–Bernstein Theorem*: if there exist injections $f\colon A \to B$ and $g\colon B \to A$, then $A$ and $B$ have the same cardinality.
   (a) Suppose $C \subset A$ and that there is an injection $f\colon A \to C$. Define $A_1 := A$, $C_1 := C$, and for $n > 1$ recursively define $A_n := f(A_{n-1})$ and $C_n := f(C_{n-1})$. Show that

   $$A_1 \supset C_1 \supset A_2 \supset C_2 \supset A_3 \supset \cdots$$

   and that $f(A_n \setminus C_n) = A_{n+1} \setminus C_{n+1}$ for all $n \in \mathbb{N}$.
   (b) Using the notation from the previous part, show that $h\colon A \to C$ defined by

   $$h(x) := \begin{cases} f(x) & \text{if } x \in A_n \setminus C_n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

   is a bijection. [**Hint:** draw a picture.]
   (c) Prove the Schröder–Bernstein Theorem.