Exercises: (Chapter 8 §1, §2, and §3)

1. Find two solutions of the initial-value problem

$$\begin{cases} x' = x^{1/3} \\ x(0) = 0 \end{cases}.$$

   Hint: Observe that the equation is separable, or try $x(t) = ct^{\lambda}$.

2. Show that both $x(t) = -t^2/4$ and $x(t) = 1 - t$ are solutions of the initial-value problem

$$\begin{cases} 2x' = \sqrt{t^2 + 4x} - t \\ x(2) = -1 \end{cases}.$$

   Why does this not contradict Theorem 2 in section 8.1 of the book about initial-value problem uniqueness?

3. Verify that the function $x(t) = t^2/4$ solves the initial-value problem

$$\begin{cases} x' = \sqrt{x} \\ x(0) = 0 \end{cases}.$$

   Apply Euler's method with $h = 0.5$ to numerically approximate $x(1)$. What happens? Explain why the numerical solution differs so much from the solution $t^2/4$ at $t = 1$. Would using a smaller value of h help?

4. Consider the initial-value problem

$$\begin{cases} (\mathrm{e}^t + 1)x' + x\mathrm{e}^t - x = 0 \\ x(0) = 3 \end{cases}.$$

   (a) Check that the analytic solution is given by $x(t) = \frac{12\mathrm{e}^t}{(1+\mathrm{e}^t)^2}$.

   (b) Write a computer program to numerically solve this initial-value problem on the interval $-2 \le t \le 0$ using Euler's method with $h = -0.01$. Plot your numerical solution against the analytic solution. What is the error between them at $t = -2$?

   (c) Now write a computer program to numerically solve this initial-value problem on the interval $-2 \le t \le 0$ using the fourth-order Runge-Kutta method with $h = -0.01$. Plot this numerical solution against the analytic solution as well. What is the error between them at $t = -2$?

   (d) Discuss your results. Which method produced a more accurate numerical solution, Euler's method or the fourth-order Runge-Kutta method? How many digits of accuracy does each numerical solution have at $t = -2$ compared to the analytic solution?