x	‖	-1	0	1
y	‖	5	7	9

Table 1: The data for Problem 1.

x	‖	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
y	‖	1	1	-1	-1

Table 2: The data for Problem 6.

Exercises: (Chapter 6 §4, §12, and Notes)

1. Find a natural cubic spline function whose knots are $t_0 = -1, t_1 = 0$, and $t_2 = 1$ and that takes on the values in Table 1.

2. By taking real and imaginary parts in a suitable complex exponential equation, prove that the following two equations hold. Let k, n be positive integers.

$$\frac{1}{n}\sum_{j=0}^{n-1}\cos\left(\frac{2\pi jk}{n}\right) = \begin{cases} 1 & \text{if } k \text{ divides } n \\ 0 & \text{otherwise} \end{cases}.$$

$$\frac{1}{n}\sum_{j=0}^{n-1}\sin\left(\frac{2\pi jk}{n}\right) = 0.$$

3. Prove the following useful identities involving complex conjugation. Let $z_1, z_2 \in \mathbb{C}$.
   (a) Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
   (b) Show that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$.
   (c) Show that $|z_1|^2 = z_1\overline{z_1}$.
   (d) Show that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}\,(z_1\overline{z_2})$.
   Here, $\text{Re}\,(a + b\mathbb{i}) = a$. That is, $\text{Re}\,(a + b\mathbb{i})$ is the *real part* of the complex number $a + b\mathbb{i}$.

4. Let $z \in \mathbb{C}$ be nonzero. Show that

$$z^{-1} := \frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

5. Prove that the normalized DFT matrix, $F \in \mathbb{C}^{n\times n}$, with entries given by

$$F_{\omega,j} = \mathbb{e}^{\frac{-2\pi\mathbb{i}\omega j}{n}}/\sqrt{n} \quad \text{for all } 0 \leq j, \omega \leq n-1$$

is unitary.

6. Find the trigonometric polynomial

$$p(x) = \widehat{p}_{-1}\mathbb{e}^{-\mathbb{i}x} + \widehat{p}_0 + \widehat{p}_1\mathbb{e}^{\mathbb{i}x} + \widehat{p}_2\mathbb{e}^{2\mathbb{i}x}$$

that interpolates the data in Table 2. Write p both in complex exponential form as above, and as a sum of sines and cosines.