Exercises: (Chapter 10 §1 – §4)

1. Show that if S and T are convex sets, then λS for $\lambda \in \mathbb{R}^+$, $S+T$, and $S-T$ are also convex. (The set $S+T$ is defined as the set of all sums $s+t$, where $s \in S$ and $t \in T$).

2. Can a bounded set have a convex complement? Justify your answer with a proof.

3. Prove that if $K \subset \mathbb{R}^n$ is a closed convex set, then every point $p \notin K$ has a unique closest point $k \in K$.

4. Convert the following problem to the standard and dual linear programming forms:

   Minimize $|x_1 + x_2 + x_3|$ subject to the constraints

$$\begin{cases} x_1 - x_2 = 5 \\ x_2 - x_3 = 7 \\ x_1 \leq 0 \\ x_3 \geq 2 \end{cases} .$$

5. **Stirling's formula** gives the following estimate of $n!$,

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

   (a) Use Stirling's formula to derive the approximate formula for the binomial coefficient

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \approx \sqrt{\frac{n}{2\pi m(n-m)}}\left(\frac{n}{n-m}\right)^n\left(\frac{n-m}{m}\right)^m .$$

   (b) Use part (a) to verify that

$$\binom{300}{100} \approx 4 \times 10^{81}.$$

6. Solve the following problems using the tableau method.

   (a) Maximize $F(x) = 6x_1 + 14x_2$ subject to the constraints

$$\begin{cases} x_1 + x_2 \leq 12 \\ 2x_1 + 3x_2 \leq 15 \\ x_1 + 7x_2 \leq 21 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases} .$$

   (b) Now change the second constraint inequality above to $2x_1 + 3x_2 \geq 15$ and then solve that altered problem.