Exercises: (Chapter 8 §6, §7, §8 and Fourier Lecture)

1. Convert the system of second-order differential equations

$$\begin{cases} x'' - x'y = 3y'x\log t \\ y'' - 2xy' = 5x'y\sin t \end{cases}$$

into a system of first-order equations in which t does not appear explicitly.

2. Write a program in the language of your choice to solve this initial-value problem using the Taylor-series method.

$$\begin{cases} x' = t + x^2 + y \\ y' = t^2 - x + y^2 \\ x(-1) = 0.43 \\ y(-1) = -0.69 \end{cases}.$$

Include terms involving h, h^2, and h^3 in your Taylor series expansions of both $x(t)$ and $y(t)$. Let $h = 0.01$ and continue the solution to $t = 1$. **Turn in both your code and a plot of x and y on $[-1, 1]$.**

3. In this exercise we will use the shooting method to analytically solve a boundary-value problem.

   (a) Solve this initial-value problem in terms of $z \in \mathbb{R}$,

$$\begin{cases} x'' = -9x \\ x(0) = 1 \\ x'(0) = z \end{cases}.$$

   Denote the solution by $x_z(t) = x(t)$.

   (b) Now use your solution form part (a) to solve the boundary-value problem

$$\begin{cases} x'' = -9x \\ x(0) = 1 \\ x(\pi/6) = 5 \end{cases}$$

   by writing the function $\phi(z) = x_z(\pi/6) - 5$ explicitly, and then solving $\phi(z) = 0$ for z. What is the resulting solution to the boundary-value problem?

4. In solving the two-point boundary-value problem

$$\begin{cases} x'' - 37t^2x' = 95 \\ x(6) = 1 \\ x(12) = 2 \end{cases}$$

using the shooting method based on the secant method, we obtain two pairs of numbers: $(z_1, x_{z_1}(12)) = (4, 5)$ and $(z_2, x_{z_2}(12)) = (2, 9)$. What initial value problem is solved for the next iteration in this procedure?

5. Use trigonometric polynomials to solve the partial differential equation

$$\frac{\partial}{\partial t}u = 5\frac{\partial^2}{\partial x^2}u - \cos(3x),$$

where $u(x, 0) = \sin(x)$.