Exercises: (Chapter 1 §1 and §2, Chapter 2 §1)

1. Show that $f(x) = x\sin(1/x)$, with $f(0) = 0$, is continuous at 0 but not differentiable at 0. Do this by using the limit definitions of continuity and differentiation from calculus and reminding yourself how they work (i.e., take the corresponding limits and see what happens!).

2. Now show that $f(x) = x^2 \sin(1/x)$, with $f(0) = 0$, is once differentiable at 0, but not twice differentiable at 0. Again, do this by considering the correct limit definitions (possibly after taking a derivative).

3. Criticize this reasoning (it's not quite right – explain why!): The function f defined by
$$f(x) := \begin{cases} x^3 + x & \text{if } x \le 0 \\ x^3 - x & \text{if } x \ge 0 \end{cases}$$
has the properties
$$\lim_{x\to 0^+} f''(x) = \lim_{x\to 0^+} 6x = 0 \quad \text{and} \quad \lim_{x\to 0^-} f''(x) = \lim_{x\to 0^-} 6x = 0.$$
Therefore, f'' is continuous at 0. What is wrong with this argument?

4. Use Taylor's Theorem with $n = 2$ to prove that the inequality $1 + x < e^x$ is valid for all real numbers except $x = 0$.

5. Prove that every sufficiently smooth function can be approximated on a closed interval of length h by a polynomial of degree n with an error that is $\mathcal{O}(h^{n+1})$ as $h \to 0$.

6. The expressions e^h and $1 + \sin(h^3)$ have the same limit as $h \to 0$. Express each in the following form with the best (i.e., largest) integer values of α and β you can.
$$f(h) = c + \mathcal{O}(h^\alpha) = c + o(h^\beta).$$

7. Let $f, g : \mathbb{R} \to \mathbb{R}$. Show that if $f = o(g)$, then $f = \mathcal{O}(g)$ also holds (as either $x \to \infty$ or $h \to 0$). Give an example showing that the converse is not true, however (as either $x \to \infty$ or $h \to 0$).

8. If $\frac{1}{10}$ is correctly rounded to the normalized binary number $(1.a_1a_2a_3\ldots a_{23})_2 \times 2^m$, what is the roundoff error? What is the relative roundoff error?

9. Compute the following relative roundoff errors.
   (a) If $\frac{3}{5}$ is correctly rounded to the binary number $(0.a_1a_2a_3\ldots a_{24})_2$, what is the relative round off error?
   (b) If $\frac{2}{7}$ is correctly rounded to the binary number $(0.a_1a_2a_3\ldots a_{24})_2$, what is the relative round off error?

10. Are these machine numbers in the "Marc-32" format (Yes or No)? Explain each answer you give.
    (a) 10^{40}
    (b) $2^{-1} + 2^{-26}$
    (c) $\frac{1}{5}$
    (d) $\frac{1}{3}$
    (e) $\frac{1}{256}$