A Few Good Problems

1. **Lagrange multipliers** *[Forget Me Not!]*: Find the triangle with maximum area that has its perimeter fixed to a given constant p. *HINT: Use Heron's formula for the area of a triangle:* $A = \sqrt{s(s-a)(s-b)(s-c)}$, *where* $s = p/2$ *is constant, and* a, b, c *are the lengths of the triangle's three sides. Find the side lengths* a, b, c. *Working with* A^2 *instead of* A *will be easier and will not change your answer.*

2. **Continuity and the Derivative** *[Forget Me Not!]*: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Hölder-continuous if there exist positive constants C and α so that $\|f(\vec{x}) - f(\vec{y})\| \leq C \|\vec{x} - \vec{y}\|^\alpha$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Describe all Hölder-continuous functions for which $\alpha > 1$. *HINT: Consider the definition of the derivative of an arbitrary Hölder-continuous function f when $\alpha > 1$.*

3. **Bijective Maps** *[From This Week!]*: Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(x, y) \;=\; T\,(u, v) = \left(\frac{\sin(u)}{\cos(v)}, \frac{\sin(v)}{\cos(u)}\right).$$

Prove that T is a bijection (i.e., is both 1−1 and onto) from the triangle $D^* = \big\{(u, v) \;\big|\; u > 0,\ v > 0,\ u + v < \pi/2\big\}$ into the unit square $D = \big\{(x, y) \;\big|\; 0 < x < 1,\ 0 < y < 1\big\}$. *HINT: For proving that T is $1 - 1$ the following trig formulas will be useful:* $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ *and* $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$. *In addition, note that* $\sin(\theta)$ *is itself* $1-1$ *for* $-\pi/2 < \theta < \pi/2$. *To prove that T is onto you can start by squaring both $x(u, v)$ and $y(u, v)$. Then, take advantage of the fact that* $\sin^2\theta + \cos^2\theta = 1$ *several times.*

4. **Change of Variables & Integration** *[From This Week]*: In the course of this problem you will calculate the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This is the famous *Riemann zeta function* evaluated at two – evaluating this sum was known as the **Basel problem** in the 1700s. Solving this problem is what first made Leonhard Euler famous!

   (a) Use the map T from Problem 3 to calculate the integral $\int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\, dx\, dy$.

   (b) Show that the even terms of the series above, $\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \ldots$, sum to $\frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (c) Use the geometric series formula, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, to prove that $\int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\, dx\, dy = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. *HINT: You may safely exchange the order of integration and summation in this case.*

   (d) Use parts $5(a) - 5(c)$ to calculate $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *HINT: Note that the infinite sum in part $5(c)$ accounts for all the odd terms in* $\sum_{n=1}^{\infty} \frac{1}{n^2}$.