Double Integration

1. **Double integrals as interated integrals (Fubini's Theorem):**
If $\iint_{[a,b]\times[c,d]} |f(x,y)| \, dA \; < \; \infty$ then

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) \, dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) \, dy.$$

Exercise 1: Calculate $\int\int_R (x^2y - xy)\,dA$ where $R = [0,2] \times [2,4]$. Check Fubini's theorem, i.e. check that the answer to the integral does not depend on the order of integration.

Exercise 2: (This is a very helpful thing for 2d integrals.)
Let $f(x,y) = h(x)g(y)$ where $h : [a,b] \to \mathbb{R}$ and $g : [c,d] \to \mathbb{R}$ are continuous functions. Show that

$$\iint_R h(x)g(y)\,dx\,dy = \left[\int_a^b h(x)\,dx \right] \left[\int_c^d g(y)dy \right] ,$$

where $R = [a,b] \times [c,d]$

2. **An Integral for which Fubini's Theorem Fails:**
Although Fubini's theorem holds for most functions met in practice, we must still exercise some caution. The next few exercises will demonstrates a case where Fubini fails.

Exercise 3: In preparation for the next problem, prove that $\frac{1}{2}\sin 2\theta = \frac{\tan\theta}{1+\tan^2\theta}$ by using that:

(a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$
(b) $1 = \cos^2\theta + \sin^2\theta$, and
(c) $\sin 2\theta = 2\cos\theta\sin\theta$.

Exercise 4: Use a substitution involving the tangent function to show that

$$\int_0^1 \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2} dy \;=\; \frac{1}{x^2} \int_0^1 \frac{1 - \left(\frac{y}{x}\right)^2}{\left(1 + \left(\frac{y}{x}\right)^2\right)^2} dy \;=\; \frac{1}{1 + x^2}.$$

You will need to use that:

(a) $1 - \tan^2\theta = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta} = \frac{\cos 2\theta}{\cos^2\theta}$,

(b) $1 + \tan^2\theta = \frac{1}{\cos^2\theta}$, and

(c) Exercise 3.

Exercise 5: Use Exercise 4 and integration formula 37 from the inside cover of your book to show that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\ dy\ dx = \frac{\pi}{4}.$$

Exercise 6: Now use Exercise 4 and/or 5 to show that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\ dx\ dy = -\frac{\pi}{4}.$$

Exercise 7: Have Exercises 5 and 6 just violated Fubini's Theorem on page 277 of the book? Why or why not? What must $\int_0^1 \int_0^1 \left| \frac{x^2-y^2}{(x^2+y^2)^2} \right|\ dx\ dy$ equal?

3. **Double Integrals over more general domains**

   Exercise 8: Sketch the region of integration, interchange the order and evaluate

$$\int_0^1 \int_{1-y}^1 (x + y^2) dx dy \,.$$

4. **Volume Below** $z = f(x, y)$

   Suppose that the function f is continuous and nonnegative on the bounded plane region R. Then the *volume* V of the solid that lies below the surface $z = f(x, y)$ and above the region R is defined to be

$$V = \iint_R f(x, y)\, dA$$

   provided that this integral exists.

5. **Area of the plane region** R

   To calculate the area of the plane region R, $a(R)$, we find the volume below the surface $f(x, y) = 1$, i.e.

$$a(R) = \iint_R 1\, dA$$

Exercise 9: Find the area between $y = x + 2$ and $y = x^2$.

Exercise 10: Find

$$\iint_D (1 - \sin(\pi x))y \, dA \,,$$

where D is the domain bounded by the x-axis and the curve $y = \cos \pi x$, and lying between $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.

Exercise 11: Sketch the region of integration, interchange the order and evaluate:

$$\int_0^{\pi/2} \int_0^{\cos\theta} \cos(\theta) \, dr d\theta \,.$$