The Cross Product

1. **Cross Product**

Definition 1 (Cross Product) *The* **cross product***, or* **vector product***, of the vectors* $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ *and* $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ *is another vector that is defined algebraically by the formula*

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle.$$

Ex. 1: Prove that the cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

TA Lecture: **Calculating the Cross Product**

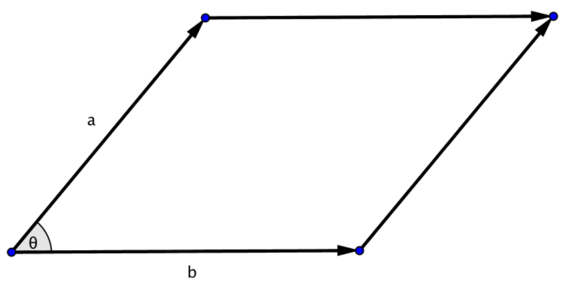
Ex. 2: Calculate the cross product of $\langle 1, 0, 0\rangle$ with $\langle 0, 1, 0\rangle$.

Ex. 3: Calculate the cross product of $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ with $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

2. **Geometric Significance**

Ex. 4: Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ (measured so that $0 \leq \theta \leq \pi$). Prove that

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta = \text{the area of the parallelogram with sides } \mathbf{a} \text{ and } \mathbf{b} \text{ (below).}$$



Ex. 5: Prove that two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = 0$.

Theorem 2 (Algebraic Properties of the Cross Product) *If* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are vectors and* k *is a real number, then*

(a) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$,

(b) $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$,

(c) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$,

(d) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$,

(e) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Definition 3 (The Scalar Triple Product) *The scalar triple product of the vectors* $\mathbf{a}, \mathbf{b}$, *and* $\mathbf{c}$ *is* $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

TA Lecture: **Calculating the Scalar Triple Product**

Ex. 6: Use Ex. 4 to prove that the volume, V, of the parallelpiped determined by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ is the absolute value of the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$; that is,

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

Ex. 7: Find the scalar triple product of $\langle 2, 0, -3\rangle, \langle 1, 1, 1\rangle$, and $\langle 0, 4, -1\rangle$.

Ex. 8: Use the scalar triple product to show that the points $A(1, -1, 2), B(2, 0, 1), C(3, 2, 0)$, and $D(5, 4, -2)$ are coplanar.