Derivatives and Planes

1. **Partial Derivatives**

Definition 1 (Partial Derivatives) *The* **partial derivatives (with respect to** x **and with respect to** y**)** *of the function* $f(x, y)$ *are the two functions defined by*

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \tag{1}$$

$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} \tag{2}$$

whenever these limits exist.

Ex. (1) Compute the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for $f(x, y) = \cos(x^2 y) + y^3$.

Theorem 2 (Componentwise Differentiation) *Suppose that*

$$\mathbf{c}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is a curve, where $x(t)$*,* $y(t)$*, and* $z(t)$ *are differentiable functions from* $\mathbb{R}$ *to* $\mathbb{R}$*. Then*

$$\mathbf{c}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

The derivative vector $\mathbf{c}'(t)$ is tangent to the curve $c(t)$ at $(x(t), y(t), z(t))$.

Ex. (2) Find a vector that is tangent to the curve

$$x(t) = t^2, \quad y(t) = t, \quad z(t) = \cos(t)$$

at the point $P\,(0, 0, 1)$.

2. **Tangent Planes**

   Let $f : \mathbb{R}^2 \to \mathbb{R}$. The **plane tangent** to the surface $z = f(x, y)$ at the point $P(a, b, f(a, b))$ is the plane through P that contains the tangent lines of both the curve

$$x(t) = t, \quad y(t) = b, \quad z(t) = f(t, b)$$

   and the curve

$$x(t) = a, \quad y(t) = t, \quad z(t) = f(a, t).$$

   at P.

Board Ex. Consider the surface $z = f(x, y) = x^2 + 2xy^2 - y^3$. It contains the point $P\,(1, 1, f(1, 1)) = P\,(1, 1, 2)$. Write the equation of the line that is tangent to the curve

$$x(t) = t, \quad y(t) = 1, \quad z(t) = f(t, 1),$$

and sketch the curve inside the graph of f.

Board Ex. Consider the surface $z = f(x, y) = x^2 + 2xy^2 - y^3$. It contains the point $P\,(1, 1, f(1, 1)) = P\,(1, 1, 2)$. Write the equation of the line that is tangent to the curve

$$x(t) = 1, \quad y(t) = t, \quad z(t) = f(1, t),$$

and sketch the curve inside the graph of f.

Board Ex. Write the equation of the plane tangent to the surface $z = f(x, y) = x^2 + 2xy^2 - y^3$ at the point $P(1, 1, 2)$.

Board Ex. Consider the tangent plane

$$-4x - y + z = -3$$

to the surface $z = f(x, y) = x^2 + 2xy^2 - y^3$ at the point $P(1, 1, 2)$ that you found in the last problem. Show that the tangent plane formula on page 110 gives the same answer, i.e., that

$$\begin{aligned} z &= f(1,1) + \left[\frac{\partial f}{\partial x}(1,1)\right](x-1) + \left[\frac{\partial f}{\partial y}(1,1)\right](y-1) \\ &= f(1,1) + \mathbf{D}f(1,1) \cdot \langle x-1, y-1 \rangle. \end{aligned}$$

Challenge Ex. Show that $\mathbf{D}f(1,1)$ from above satisfies the definition of the derivative of $f(x,y) = x^2 + 2xy^2 - y^3$ at $(1,1)$ on page 111 by using the polar coordinate transform

$$x = r\cos\theta + 1,\ \ y = r\sin\theta + 1$$

in order to show that

$$\lim_{(x,y)\to(1,1)} \frac{|f(x,y) - f(1,1) - \mathbf{D}f(1,1)\cdot\langle x-1, y-1\rangle|}{\|(x,y)-(1,1)\|} = 0.$$