Vectors in $\mathbb{R}^2$

1. **Vectors in $\mathbb{R}^2$**

Definition 1 (Vector) *A* **vector** $\mathbf{v} = \langle a, b \rangle$ *in* $\mathbb{R}^2$ *is an ordered pair of real numbers. We call* a *and* b *the* ***components*** *of the vector* $\mathbf{v}$*.*

Vectors are geometrically represented by directed line segments in the cartesian plane:

Definition 2 (Equality) *The two vectors* $\mathbf{u} = \langle u_1, u_2 \rangle$ *and* $\mathbf{v} = \langle v_1, v_2 \rangle$ *are* **equal** *if* $u_1 = v_1$ *and* $u_2 = v_2$*.*

Definition 3 (Addition) *The* **sum** *of two vectors* $\mathbf{u} = \langle u_1, u_2 \rangle$ *and* $\mathbf{v} = \langle v_1, v_2 \rangle$ *is the vector*

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle.$$

Definition 4 (Scalar Multiplication) *If* $\mathbf{u} = \langle u_1, u_2 \rangle$ *and* c *is a real number, then the* **scalar multiple** $c\mathbf{u}$ *is the vector*

$$c\mathbf{u} = \langle cu_1, cu_2 \rangle.$$

Ex. If $\mathbf{u} = \langle 3, 5 \rangle$ and $\mathbf{v} = \langle -4, 4 \rangle$, find $\mathbf{u} + \mathbf{v}$, $2\mathbf{u}$, and $\mathbf{u} - \mathbf{v}$ (that is, $\mathbf{u} + (-1)\mathbf{v}$).

Definition 5 (Length) *The* **length** *of* $\mathbf{v} = \langle a, b \rangle$ *is denoted* $|\mathbf{v}|$ *and is defined as*

$$|\mathbf{v}| = |\langle a, b \rangle| = \sqrt{a^2 + b^2}.$$

Ex. Find the length of $\mathbf{v} = \langle 3, 5 \rangle$

Definition 6 (Unit Vectors) *A* **unit** *vector is a vector of length 1. If* $\mathbf{a} = \langle a_1, a_2 \rangle \neq 0$*, then*

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

is the unit vector in the same direction as $\mathbf{a}$*.*

Ex. Find a unit vector in the same direction as $\langle 3, 5\rangle$.

Ex. Two unit vectors, $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$, are often used as an alternative way to represent vectors. Write $\langle 3, 5\rangle$ as a sum of $\mathbf{i}$ and $\mathbf{j}$.

Board Ex. Show that the line segment joining the midpoints of two sides of a triangle is parallel to and half the length of its third side.

2. **Vectors in $\mathbb{R}^3$**

Definition 7 (Vector) *A* **vector** $\mathbf{v} = \langle a, b, c\rangle$ *in* $\mathbb{R}^3$ *is an ordered triple of real numbers.*

Vectors in $\mathbb{R}^3$ are geometrically represented by directed line segments in three dimensional Euclidean space. Addition, scalar multiplication, length, and unit vectors are all defined in the same way as for vectors in $\mathbb{R}^2$, but now we have *three* basic unit vectors, $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$.

Definition 8 (Distance) *The* ***distance*** *between two points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *is*

$$|PQ| = |\overrightarrow{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Ex. Find the distance between the points $A(1, 2, 3)$ and $B(3, -2, 5)$.

Definition 9 (Dot Product) *The* **dot product** *of the two vectors*

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \text{ and } \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

is the scalar defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Ex. Calculate the dot product of $\langle 1, 2, 10\rangle$ and $\langle -2, 3, 4\rangle$.

The following properties of the dot product are useful for working with the dot product.

(i) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

(ii) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(iii) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(iv) $(r\mathbf{a}) \cdot \mathbf{b} = r(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (r\mathbf{b})$

Does $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ make sense? What about $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$?

3. **Interpretation of the Dot Product**

Theorem 10 *If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, then*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$

Board Ex. Prove Thm 10:

Corollary 11 *The two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Ex. Show that the vectors $3\mathbf{i} + 5\mathbf{j}$ and $-4\mathbf{i} + 4\mathbf{j}$ are not perpendicular using the dot product.

Ex. For what value k would the vectors $\langle 3, 5, 1\rangle$ and $\langle -4, 4, k\rangle$ be perpendicular?

4. **Projections**

Board Ex. Given $\mathbf{a} = \langle 4, -5, 3\rangle$ and $\mathbf{b} = \langle 2, 1, -2\rangle$, find the projection of $\mathbf{a}$ in the direction of $\mathbf{b}$.

5. **Planes**

   How can we define a plane in space using the dot product if we are given a point $P_0(x_0, y_0, z_0)$ on the plane and a direction $\mathbf{n}$ perpendicular to the plane?

Board Ex. Given a point $(2, 3, 1)$ and a plane $2x + 3y - z = 5$, find the distance from the point to the plane.