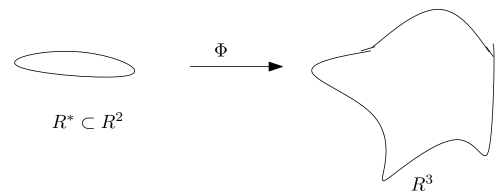
Integration on Manifolds

Definition 1 (A Simple n-dimensional Manifold) *Consider a $\mathcal{C}^2$ and $1-1$ function $\Phi : \mathbb{R}^n \to \mathbb{R}^N$ for any $N \geq n$, with $\Phi = (\Phi_1, \cdots, \Phi_N)$, where $\Phi_j : \mathbb{R}^n \to \mathbb{R}$, $\forall j = 1, \ldots, N$. Suppose that the domain of Φ is a "regular region" $R^* \subset \mathbb{R}^n$ ("regular" can mean here, e.g., that the boundary of R^* is $\mathcal{C}^2$, and that R^* is convex). We will call $\Phi(R^*) \subset \mathbb{R}^N$ a simple n-dimensional submanifold of $\mathbb{R}^N$.*

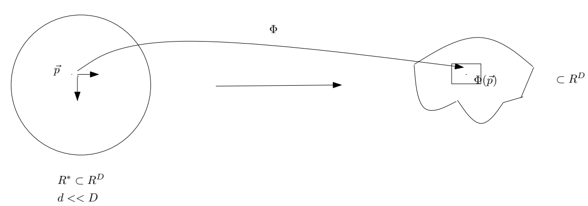
e.g.



Definition 2 *Recall the derivative of $\Phi : \mathbb{R}^n \to \mathbb{R}^N$ at $\vec{p} \in R^*$ is*

$$D\Phi|_{\vec{p}} := \begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1}(\vec{p}) & \frac{\partial \Phi_1}{\partial x_2}(\vec{p}) & \cdots & \frac{\partial \Phi_1}{\partial x_n}(\vec{p}) \\ \frac{\partial \Phi_2}{\partial x_1}(\vec{p}) & \frac{\partial \Phi_2}{\partial x_2}(\vec{p}) & \cdots & \frac{\partial \Phi_2}{\partial x_n}(\vec{p}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \Phi_N}{\partial x_1}(\vec{p}) & \frac{\partial \Phi_N}{\partial x_2}(\vec{p}) & \cdots & \frac{\partial \Phi_N}{\partial x_n}(\vec{p}) \end{pmatrix} \in \mathbb{R}^{N \times n}$$

The columns of $D\Phi|_{\vec{p}}$ span the tangent space to the n-dimensional manifold $\Phi(R^*)$ at $\Phi(\vec{p})$. These are exactly the tangent vectors to the n curves we get by holding all but one of the entries of Φ constant, as we did with surfaces when $n = 2$ and $N = 3$.



Note: The column span$\{D\Phi|_{\vec{p}}\}$ is an n-dim subspace and the affine subspace parallel to it passing through $\Phi(\vec{p})$ is tangent to $\Phi(R^*)$ at $\Phi(\vec{p})$.

Definition 3 *The n-dimensional volume element of $\Phi(R^*)$ is*

$$dV := \sqrt{\left|\det\left((D\Phi)^T (D\Phi)\right)\right|} \;\; dx_1 \; dx_2 \; \ldots \; dx_n$$

Here, $(D\Phi)^T \in \mathbb{R}^{n \times N}$ is just the usual transpose of the derivative matrix $D\Phi \in \mathbb{R}^{N \times n}$ obtained by making the i^{th}-row of D into the i^{th}-column of D^T. Note that $(D\Phi)^T (D\Phi) \in \mathbb{R}^{n \times n}$ is symmetric and square.

Definition 4 *The d-dimensional volume of $\Phi(R^*)$ is defined to be*

$$\int_{\Phi(R^*)} dV = \int_{R^*} \sqrt{\left|\det\left((D\Phi)^T (D\Phi)\right)\right|} dx_1 \; dx_2 \; \ldots \; dx_n$$

THIS SINGLE FORMULA GENERALIZES EVERYTHING WE HAVE LEARNED SO FAR ABOUT INTEGRATION ON CURVES AND SURFACES, AS WELL AS ABOUT CHANGES OF VARIABLES! The purpose of this lab will be to convince ourselves of this...

1. Exercise: Consider the curve, or one-dimensional manifold, $\mathbf{c}([0,1])$, given by a $\mathcal{C}^1$ function $\mathbf{c} : \mathbb{R} \to \mathbb{R}^3$. In this case we have
$$D\mathbf{c}|_t = \mathbf{c}\,'(t) = \begin{pmatrix} c_1'(t) \\ c_2'(t) \\ c_3'(t) \end{pmatrix} \in \mathbb{R}^{3\times 1}.$$
Show that Definition 3 agrees with our previous definition for ds in this case.

2. Exercise: Consider the two-dimensional parameterization of a region in $\mathbb{R}^2$ given by the $\mathcal{C}^2$ change of variables, $(x, y) = \Phi(u, v)$, given by $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. Show that Definition 3 agrees with our previous definition for the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$ in this case. *Hint:* $\det(AB) = \det(A)\det(B)$ *holds for all* $A, B \in \mathbb{R}^{n \times n}$.

3. Consider any surface, or two-dimensional manifold, given by a parameterization of the form $\Phi(u, v) = (u, v, f(u, v))$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^2$-function. Show that Definition 3 agrees with our previous definition for dS in this case.

4. Compute the volume of the 3-dimensional manifold in $\mathbb{R}^5$, $\Phi\left([0,1]^3\right)$, parameterized by $\Phi(u,v,t) = (u,v,u,v,t^2)$.

5. Integrate $f : \mathbb{R}^5 \to \mathbb{R}$ over the 3-dimensional manifold from the last problem, when $f(a,b,c,d,e) = a - c + b - d + \sqrt{e}$. *Note: Before you can do this, you should decide what "integrating f over $\Phi\left([0,1]^3\right)$" means!*

6. Can you find two vectors that are perpendicular to the three-dimensional tangent space to $\Phi\left([0,1]^3\right)$ at $\Phi(0,0,1)$?