Path Integrals, Curvature, and Conservative Vector Fields

Definition 1 (Unit Tangent Vector) *A curve's* **unit tangent vector** *at the point* $\mathbf{c}(t)$ *is:*

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|} \tag{1}$$

Definition 2 (Unit Normal Vector) *A curve's* **principle unit normal vector** *at* $\mathbf{c}(t)$ *is*

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \tag{2}$$

provided $\frac{d\mathbf{T}}{dt}(t) \neq \mathbf{0}$. *This vector points in the direction the curve is bending.*

1. **Review Exercise:** Show that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0 \ \forall t$ by differentiating both sides of $\|\mathbf{T}(t)\|^2 = 1$.

Definition 3 (Curvature) *The* **curvature** *of a curve C is*

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| \tag{3}$$

where $\mathbf{T}$ *is the unit tangent vector, and s is the arc length distance from the beginning of C.*

Note: If $\mathbf{c} : [0, a] \to \mathbb{R}^2$ is a unit speed parameterization of C, then $s := \int_0^s \|\mathbf{c}'(t)\| dt = \int_0^s \; dt$. Otherwise, **if $\mathbf{c} : [0, a] \to \mathbb{R}^2$ is not a unit speed parameterization of** C, then the arc length distance along the curve that one has traveled at time t is $s(t) := \int_0^t \|\mathbf{c}'(u)\| du$.

2. **Exercise – First Curvature Formula:** Prove that $\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{c}'(t)\|}$ when given a parameterization $\mathbf{c} : [0, a] \to \mathbb{R}^2$ of a given curve C.

3. **Exercise:** Suppose we are given a parameterization $\mathbf{c} : [0, a] \to \mathbb{R}^2$ of a given curve C. Write the acceleration $\mathbf{c}''(t)$ in terms of C's curvature $\kappa(t)$, scalar acceleration $a(t) := \frac{d}{dt}\|\mathbf{c}'(t)\|$, and the moving orthonormal basis vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ by computing

$$\mathbf{c}''(t) = \frac{d}{dt}\mathbf{c}'(t) = \frac{d}{dt}\left(\|\mathbf{c}'(t)\| \cdot \mathbf{T}(t)\right).$$

4. **Exercise:** Use the last exercise to derive this additional formula for curvature.

$$\kappa(t) = \frac{\|\mathbf{c}'(t) \times \mathbf{c}''(t)\|}{\|\mathbf{c}'(t)\|^3} \tag{4}$$

5. **Exercise** Find the curvature formula for the space curve $\mathbf{c} = (t, t^2/2, t^3/3)$ at a general point, at $(0, 0, 0)$, and at $(1, \frac{1}{2}, \frac{1}{3})$.

Theorem 4 (The Fundamental Theorem for Line Integrals) *Let f be a function of two or three variables and let C be a smooth curve from A to B parameterized by the vector-valued function $\mathbf{r}(t)$ for $a \leq t \leq b$. If f is continuously differentiable at each point of C, then*

$$\int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = f(B) - f(A). \tag{5}$$

6. **Note:** Gradient vector fields have integrals which only depend on the endpoints of a given path! Actually, this is the only case!

Theorem 5 *The line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ of the continuous vector field $\mathbf{F}$ is independent of path in the plane or space region D if and only if $\mathbf{F} = \boldsymbol{\nabla} f$ for some function f defined on D.*

7. This makes gradient vector fields very important – important enough to have two names...

Definition 6 *The vector field $\mathbf{F}$ defined on a region D is* **conservative** *provided that there exists a scalar function f defined on D such that*

$$\mathbf{F} = \boldsymbol{\nabla} f$$

at each point of D. In this case f is called a **potential function** *for the vector field $\mathbf{F}$.*

8. **Exercise:** Find a potential function for the conservative vector field $\mathbf{F} = \langle 2x + 5y, 5x + e^y \rangle$.

Theorem 7 *Suppose that the vector field* $\mathbf{F} = \langle P, Q \rangle$ *is continuously differentiable in an open rectangle* R *in the xy-plane. Then* $\mathbf{F}$ *is conservative in* R *– and hence has a potential function* $f(x, y)$ *defined on* R*– if and only if, at each point of* R*,*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

9. **Exercise:** Check if the following fields are conservative and if so find their potential functions.

   (a) $\langle 2xy - y^2, x^2 - 2xy \rangle$.

   (b) $\langle \cos(x)\sin(y), \sin(x)\cos(y) \rangle$.

   (c) $\dfrac{\mathbf{r}}{r^3}$, where $\mathbf{r} = \langle x, y, z \rangle$.