CMSE 823, **Test 1** **September 29, 2016**

Name: ______________________________

Instructions:

- You only need a pen, or a pencil, your text book, notes, and an eraser.
- In particular, this means no calculator or any electronic devices – **only paper!**
- You must show appropriate legible work and justify your answer to receive full credit.
- There are 100 possible points. Point values for each problem are as indicated.
- Check and make sure there are four total pages including the cover page, when you begin the exam.

Good Luck!

Problem	**Score**	**Out of**
1		**26**
2		**10**
3		**22**
4		**14**
5		**14**
6		**14**
Total		**100**

1. **The Singular Value Decomposition:** Find the singular value decomposition of $A \in \mathbb{R}^{3\times 2}$ below, and then answer the related questions. *[26 points]*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) $\|A\|_2 = \max_{\mathbf{x}\in\mathbb{C}^2\setminus\{\mathbf{0}\}} \frac{\|A\mathbf{x}\|_2}{\|x\|_2} =$ ______________________________ .

(b) $\|A\|_{\mathrm{F}} =$ ______________________________.

(c) $\|A\|_1 =$ ______________________________.

(d) $\|A\|_\infty =$ ______________________________.

(e) An orthonormal basis for the column space of A is:

(f) An orthonormal basis for the column space of A^* is:

(g) An orthonormal basis for the null space of A is:

(h) An orthonormal basis for the null space of A^* is:

Extra work space for problem 1

2. **Projections and Least Squares:** Given $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -3 \\ 0 & 0 \\ 1 & -3 \end{pmatrix}$.

(a) Find $P_{C(A)}\mathbf{b}$ = the orthogonal projection of $\mathbf{b}$ onto the column space of A. *[6 points]*

(b) Find the error vector $\mathbf{e} = \mathbf{b} - P_{C(A)}\mathbf{b}$. Check that $\mathbf{e}$ is perpendicular to $C(A)$. *[2 points]*

(c) Find $\mathbf{x} = \arg\min_{\mathbf{y}\in\mathbb{R}^2} \|A\mathbf{y} - \mathbf{b}\|_2$. *[2 points]*

3. **The QR decomposition:** Let

$$A = \begin{pmatrix} 10 & 9 \\ 20 & -15 \\ 20 & -12 \end{pmatrix}.$$

(a) Construct a matrix $\hat{Q}$ with orthonormal columns, and an upper triangular matrix R, such that $A = \hat{Q}R$. *[16 points]*

(b) Find $\mathbf{x} = \arg\min_{\mathbf{y}\in\mathbb{R}^2} \|A\mathbf{y} - 45\mathbf{e}_2\|_2$. *[6 points]*

4. **Householder Reflectors:** Let $F \in \mathbb{C}^{n \times n}$ be a Householder reflector for a nonzero $\mathbf{x} \in \mathbb{C}^n$. Suppose that $F\mathbf{x} = \alpha \|\mathbf{x}\|_2 \mathbf{e}_1$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Compute all the eigenvalues of F. Make sure to justify your answer fully. *[14 points]*

5. **Properties of Norms:** Let $A \in \mathbb{C}^{n\times n}$ be a matrix whose induced ℓ_2-norm is less than 1 (i.e., with $\|A\|_2 < 1$). Prove that $I - A$ is invertible, where I is the $n \times n$ identity matrix. *Hint: Argue that* $(I - A)\mathbf{v} \neq \mathbf{0}$ *for any nonzero vector* $\mathbf{v} \in \mathbb{C}^n$. *[14 points]*

6. **More on the SVD:** Suppose that $A \in \mathbb{C}^{m \times n}$, $m \geq n$, has the block form

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where $A_1 \in \mathbb{C}^{n \times n}$ is invertible, and $A_2 \in \mathbb{C}^{(m-n) \times n}$ is arbitrary. Show that the smallest singular value of A is $\geq$ the smallest singular value of $A_1 > 0$. *[14 points]*