1 Krein-Milman theorem

We are going to prove a following wonderful theorem

Theorem 1.1. *Let X be a locally convex linear toplogical vector space. Let A be a convex compact in X. Then*

- *The set of extrem points is not empty.*
- *A is a closure of the convex hull of its extrem epoints.*

Proof. Firstly we should mention what is the face of a convex set.

Definition 1.2. A nonemty set F is a face of A if whenever $\alpha x + (1-\alpha)y = z \in F$, for some $0 \leq \alpha \leq 1$ and $x, y \in A$ then $x, y \in F$.

We need a following auxiliary lemma:

Lemma 1.3. *Take any element $\ell \in X^*$ (continuous linear functional). We claim that a set $F_\ell = \{y \in A : \ell(y) = \max_{x\in A} \ell(x)\}$ is a face.*

Proof. Firstly, note that since A is compact and ℓ continuous therefore F_ℓ is nonempty. Suppose $\alpha x + (1-\alpha)y = z \in F_\ell$. Then

$$\max_{x\in A} \ell(x) = \ell(z) = \alpha\ell(x) + (1-\alpha)\ell(y) \leq \alpha \max_{x\in A} \ell(x) + \max_{x\in A} \ell(x) \leq \max_{x\in A} \ell(x).$$

On the other hand we always have the equality which can be fullfield if and only if $\ell(y) = \max_{x\in A} \ell(x)$ and $\ell(x) = \max_{x\in A} \ell(x)$. Thus $x, y \in F_\ell$.

□

Now we return to the proof of the first part of the theorem. By Corns lemma we do the following procedure: if A consists with only one point then we are done. If there are two distinct points say $x \neq y$ both belonging to A, then by hahn-banach theorem there exists $\ell \in X^*$ which strictly separates these two points. In other words $\ell(x) > \ell(y)$. Now we construct the face F_ℓ surely it does not contain the point y. Then we look at F_ℓ and make the same procedure. Thus we obtain the sequence of faces $\{F_\ell\}$. It is linearly ordered (ordered by inclusion) set. They are compact (As a closed (indeed) subset of compact set) so they have an *upper* bound, for example intersecion of compacts is not empty. We choose the minimal element. Note that a minimal element is a face (easy). If it contains more that 1 point then we make the same procedure which will bring us to the contradiction with minimality. Thus we obtain the extreme point.

Now we are ready to prove the second part of the theorem. Let E be a set of extreme points in A. Let $\operatorname{CConv} E$ be a closure of its convex hull.

Firstly note that $\operatorname{CConv} E \subseteq A$. We need to prove the convers inclusion. From contrary, let $x \in A \setminus \operatorname{CConv} E$. Then we use the hahn-banach theorem to the point x (as a compact set) and closed convex set $\operatorname{CConv} E$. There exists $\ell \in x^*$ such that we have $\sup_{y \in \operatorname{CConv} E} \ell(y) < \ell(y)$. Then we constract the face F_ℓ. Surely it does not intersect the set $\operatorname{CConv} E$ and by the first part of the theorem it has an extreme point. So we obtain the contradiction.

$\square$