Definition of a vector space

A *vector space* is a set V, whose elements are called *vectors* endowed with

- a rule for addition that associates to each pair $\mathbf{x}, \mathbf{y} \in V$ an element $\mathbf{x} + \mathbf{y} \in V$, and
- a rule for scalar multiplication that associates to each $\mathbf{x} \in V$ and $r \in \mathbb{R}$ an element $r\mathbf{x} \in V$,

such that, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{R}$,

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{y} + (\mathbf{x} + \mathbf{z})$

A3. $\exists$ a vector $\mathbf{0} \in V$ s.t. $\mathbf{x} + \mathbf{0} = \mathbf{x}$

A4. For each $\mathbf{x} \in V$, $\exists$ an "opposite vector" $-\mathbf{x} \in V$ s.t. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

A5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

A6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

A7. $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$

A8. $1 \cdot \mathbf{x} = \mathbf{x}$.

Notes (a) These axioms implicitly assume that the properties of the real numbers, of sets, and of the symbol $=$ (e.g. adding the same thing to both sides preserves equality) are known and will be used freely. Thus in proofs you will have occasion to use the abbreviations

$\mathbb{R}$ Prop. Set Prop. $=$ Prop.

for "property of the real numbers" , "property of sets" and "property of equality".

(b) When checking if a given set is a vector space, the two bulleted requirements, and axioms A3 and A4, are the most important.

Consequences of the axioms. From the axioms, one can derive numerous simple consequences that are useful in calculations. Each is proved from the axioms *and from previously proved facts.* Once a fact is proved, it gets added to our basket of "known facts" and can be used from then on.

Here are several examples. Each is called a "Lemma", and a proof is given 2-column format.

Lemma 1. $\mathbf{0} + \mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in V$. (Note that 0 is a number, but $\mathbf{0}$ (also written $\vec{\mathbf{0}}$) is a vector.

Proof:

$$\begin{aligned} \mathbf{0} + \mathbf{v} &= \mathbf{v} + \mathbf{0} && \textbf{A1} \\ &= \mathbf{v} && \textbf{A3} \quad \square \end{aligned}$$

Lemma 2. $\mathbf{v} + \mathbf{v} = 2\mathbf{v} \qquad \forall \mathbf{v} \in V$.

Proof:

$$\begin{aligned} \mathbf{v} + \mathbf{v} &= 1 \cdot \mathbf{v} + 1 \cdot \mathbf{v} && \textbf{A8} \\ &= (1+1)\mathbf{v} && \textbf{A6} \\ &= 2\mathbf{v} && \mathbb{R}\text{ Prop.} \quad \square \end{aligned}$$

Lemma 3. The additive inverse $\mathbf{0} \in V$ is unique.

Proof: If $\mathbf{0}$ and $\mathbf{0}'$ both satisfy A3, then

$$\begin{aligned} \mathbf{0}' &= \mathbf{0}' + \mathbf{0} && \text{A3 for } \mathbf{0} \\ &= \mathbf{0} + \mathbf{0}' && \text{A1} \\ &= \mathbf{0} && \text{A3 for } \mathbf{0}'. \quad \square \end{aligned}$$

Lemma 4. (a) $0 \cdot \mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$, and

(b) $\alpha \cdot \mathbf{0} = \mathbf{0}$ for all $\alpha \in \mathbb{R}$.

(c) If $\alpha\mathbf{v} = \mathbf{0}$ then $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$.

Proof: (a) Proved in class.

(b)
$$\begin{aligned} \mathbf{0} &= \alpha\mathbf{0} + -(\alpha\mathbf{0}) && \underline{\qquad} \\ &= \alpha(\mathbf{0} + \mathbf{0}) + -(\alpha\mathbf{0}) \\ &= (\alpha\mathbf{0} + \alpha\mathbf{0}) + -(\alpha\mathbf{0}) \\ &= \alpha\mathbf{0} + \big(\alpha\mathbf{0} + -(\alpha\mathbf{0})\big) \\ &= \alpha\mathbf{0} + \mathbf{0} \\ &= \alpha\mathbf{0} && \square \end{aligned}$$

(c) If $\alpha \neq 0$ then its inverse $\frac{1}{\alpha}$ exists. Then

$$\begin{aligned} \mathbf{v} &= 1 \cdot \mathbf{v} && \underline{\qquad} \\ &= \vdots \\ &= \mathbf{0} && \square \end{aligned}$$

Lemma 5. (Cancellation Law) If $\mathbf{x} + \mathbf{v} = \mathbf{y} + \mathbf{v}$, then $\mathbf{x} = \mathbf{y}$. Done in class.

Background: Axiomatic Systems

Most areas of modern mathematics are organized axiomatically. This is the approach that Euclid took to developing geometry: he began with 5 axioms (Euclid's "postulates") and built the entire subject from there. In the 19th and early 20th centuries this scheme was applied to other areas of mathematics and refined into what is known as the *axiomatic system.* In particular, linear algebra is organized as an axiomatic system.

There are four parts of an axiomatic system.

1. Terms and definitions. Each axiomatic system begins with a few *undefined terms.* The undefined terms in linear algebra are *point*, *vector*, *set*, *element*, and a few others. After that, new terms are introduced with precise definitions.

2. Axioms. An *axiom* or *postulate* (the words are interchangeable) is a statement that is accepted without proof. The subject (linear algebra for us) begins with a short list of axioms. Everything else is logically derived from them.

Sometimes axioms refer to other mathematical subjects or objects. The axioms of linear algebra assume that you are familiar with the properties of the real numbers, and of sets (which themselves can be developed as axiomatic systems).

3. Theorems. The largest part, by far, of an axiomatic system consists of theorems and their proofs. A *theorem* is a "if–then" statement that has been proved to be a logical consequence of the axioms. Vocabulary: the words *theorem* and *proposition* are synonyms, a *lemma* is a theorem that is stated as a step toward some more important result, and a *corollary* is a theorem that can be quickly and easily deduced from a previously-proved theorem.

Theorems are organized in a strict logical order, building on each other. The first theorem is proved using only the axioms. The second theorem is proved using only the first theorem and the axioms, etc..

4. Models. An axiomatic system is a purely logical construction. The human brain is not good at understanding complicated logical constructions. Thus it is advantageous to have a way of thinking about or visualizing the undefined terms that is compatible with human experience.

An *interpretation* of an axiomatic system is a particular way of giving meaning to the undefined terms in that system. An interpretation is called a *model* if the axioms are true statements in that interpretation. As a result, all of the theorems are also true for the model.

A good model makes it possible to visualize and guess theorems, and it provides guidance in developing proofs. For linear algebra, the best model is the pictures of vector addition by parallelograms and scalar multiplication by rescaling vectors described on Day 1.

Vector Space Axioms Homework: Give proofs, similar to those of Lemmas 1-4 above, that each of the following facts holds for all $\mathbf{x}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{R}$.

Thus you should prove Lemmas 6-12. For each, you are allowed to use the axioms and all previously proved Lemmas (Lemmas 1-5 can be used to prove Lemma 6, Lemmas 1-6 used to prove Lemma 7, etc.).

- **Lemma 6. (Uniqueness of Additive inverse)** If $\mathbf{v} + \mathbf{x} = \mathbf{0}$ then $\mathbf{x} = -\mathbf{v}$.

  Start with $\mathbf{x} = \mathbf{x} + \mathbf{0} = \cdots$.

- **Lemma** 7. $(-1)\mathbf{v} = -\mathbf{v}$

  Start with the words "Let $\mathbf{x} = (-1)\mathbf{v}$*". Then show that* $\mathbf{v} + \mathbf{x} = \mathbf{0}$ *and quote Lemma 6.*

- **Lemma** 8. If $\mathbf{v} = -\mathbf{v}$ then $\mathbf{v} = \mathbf{0}$.

- **Lemma 9.** $-(-\mathbf{v}) = \mathbf{v}$.

- **Lemma 10.** $\alpha(-\mathbf{v}) = -(\alpha\mathbf{v})$.

- **Lemma 11.** $(-\alpha)\mathbf{v} = -(\alpha\mathbf{v})$.

- **Lemma 12.** If $\mathbf{v} \neq \mathbf{0}$ and $\alpha\mathbf{v} = \beta\mathbf{v}$, then $\alpha = \beta$.

  Start with $(\alpha - \beta)\mathbf{v} = \big(\alpha + (-\beta)\big)\mathbf{v}$ *and use Lemmas 4a and 4c.*