Chapter 1

Complex Numbers and Functions

1.1 Definition

We have learned several systems of numbers. They are

1. $\mathbb{N} = \{1, 2, 3, \dots\}$, natural numbers.

2. $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$, integers.

3. $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\}$, rational numbers.

4. $\mathbb{R} = \{\text{rationals and irrationals}\}$, real numbers.

Their relations are $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Among them $\mathbb{Q}$ and $\mathbb{R}$ are called fields. This means that there are operations in these sets $a + b$, $a - b$, $a \cdot b$, a/b (if $b \neq 0$), and there are special numbers 0 and 1 such that $0 + a = a$ and $1 \cdot a = a$.

We will learn a new field of numbers:

5 $\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}$, complex numbers,

which extends $\mathbb{R}$. The symbol i stands for a square root of -1, i.e.,

$$i^2 = -1.$$

Since $x^2 \geq 0$ for every $x \in \mathbb{R}$, i cannot be a real number. For $z = x + yi \in \mathbb{C}$, x is called the real part of z, and y is called the imaginary part of z. We write $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

There are several reasons why we need complex numbers

1. Fundamental theorem of algebra: every nonconstant complex polynomial has at least one complex root. This is not true for real polynomial, e.g., $x^2 + 1 = 0$ has no real root.

2. Linear algebra: every complex matrix has a complex eigenvalue.

3. The properties of familiar functions such as polynomials, e^x, $\log(x)$, $\sin(x)$, $\cos(x)$.

4. Compute definite integral and series such as $\int_0^\infty \frac{\sin(x)}{x}$ and $\sum_{n=1}^\infty \frac{1}{n^2}$.

5. Harmonic functions in two dimensions.

6. The definition of Fourier transform.

7. The oscillation phenomena in differential equations.

We identify $x + 0i \in \mathbb{C}$ with $x \in \mathbb{R}$. In this way, we have

$$\mathbb{R} \subset \mathbb{C}.$$

The operations $+$ and $\cdot$ on $\mathbb{C}$ are defined as follows.

1. $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$.

2. $(x_1 + y_1 i) \cdot (x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$.

To understand the product formula, we may expand it and use $i^2 = -1$:

$$(a_1 + a_2 i) \cdot (b_1 + b_2 i) = a_1 b_1 + a_1 b_2 i + a_2 i b_1 + a_2 i b_2 i = (a_1 b_1 - a_2 b_2) + (a_1 b_2 + a_2 b_1)i.$$

Note that when $x_1 + y_1 i \in \mathbb{R}$, i.e., $y_1 = 0$, we have a scalar product

$$x_1(x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i.$$

The summation and multiplication satisfy commutative law, associative law, and distributive law:

1. $z_1 + z_2 = z_2 + z_1$, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$;

2. $z_1 z_2 = z_2 z_1$, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$;

3. $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$.

Because of the commutative law, the complex number $x + yi$ may also be written as $x + iy$ or $yi + x$. The two complex numbers $0, 1 \in \mathbb{R} \subset \mathbb{C}$ are special in the sense that $0 + z = z$ and $1z = z$. For $z = x + yi$, $-z := (-x) + (-y)i = (-1)z$. Then $z + (-z) = 0$. The subtraction is defined to be $z - w = z + (-w)$.

Let $z = x + yi \in \mathbb{C}$. We define $\overline{z}$ to be $x - yi$, i.e., $x + (-y)i$. It is called the conjugate of z. There are several simple properties:

1. $\overline{\overline{z}} = z$;

2. $\overline{z} = z$ if and only if $z \in \mathbb{R}$;

3. $z + \overline{z} = 2 \operatorname{Re} z$, $z - \overline{z} = i2 \operatorname{Im} z$;

4. $\overline{z \pm w} = \overline{z} \pm \overline{w}$;

5. $\overline{zw} = \overline{z} \cdot \overline{w}$.

6. For $z = x + yi$, $z\overline{z} = x^2 + y^2$.

For $z = x + yi \in \mathbb{C}$, the absolute value of z is

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}.$$

So we have $|0| = 0$, $|z| > 0$ if $z \neq 0$. A useful formula is $z\overline{z} = |z|^2$. Thus,

$$|zw|^2 = (zw)\overline{zw} = zw\overline{z}\overline{w} = |z|^2|w|^2.$$

Taking square root, we get

$$|zw| = |z||w|.$$

Since $-z = (-1)z$, we get $|-z| = |-1||z| = |z|$. Since $x^2, y^2 \leq x^2 + y^2$, taking square root, we get

$$|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z|.$$

Because $(-y)^2 = y^2$, we have $|\overline{z}| = |z|$.

To define the division, we first need to find $z^{-1} = 1/z = \frac{1}{z}$ with $zz^{-1} = 1$ for every $z \in \mathbb{C} \setminus \{0\}$. Since $z\overline{z} = |z|^2 > 0$, we see that

$$z^{-1} = \frac{1}{|z|^2}\overline{z} = \frac{1}{x^2 + y^2}(x - yi) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

The division $z/w = \frac{z}{w}$ is defined to be zw^{-1} for $w \neq 0$. Since $z = (z/w)w$, we have $|\frac{z}{w}| = \frac{|z|}{|w|}$. Now we define the integer power of a complex number. First, let $z^0 = 1$ for all $z \in \mathbb{C}$. This includes $0^0 = 1$. Second, for $n \in \mathbb{N}$, let $z^n = \underbrace{z \cdots z}_{n}$ for all $z \in \mathbb{C}$. Third, for $n \in \mathbb{Z}$ with $n < 0$, let $z^n = \frac{1}{z^{|n|}}$ be defined for $z \in \mathbb{C} \setminus \{0\}$. It is clear that $|z^n| = |z|^n$ whenever z^n is defined.

Now we come to the geometry. We may identify each complex number $z = x + yi$ with a point with coordinate (x, y) in the plane. Then the x-axis is composed of real numbers, and the y-axis is composed of pure imaginary numbers: iy, $y \in \mathbb{R}$. The conjugate is a reflection about the x-axis. The addition follows the parallelogram rule. The distance between $z = x + yi$ and 0 is $\sqrt{(x-0)^2 + (y-0)^2} = |z|$. The distance between $z, w \in \mathbb{C}$ is then $|z - w|$. From plane geometry, we have the triangle inequality:

$$|z + w| \leq |z| + |w|.$$

Now we give a new proof using complex numbers:

$$\begin{aligned}
|z+w|^2 &= (z+w)\overline{z+w} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + w\overline{w} + w\overline{z} + z\overline{w} \\
&= |z|^2 + |w|^2 + w\overline{z} + \overline{w\overline{z}} = |z|^2 + |w|^2 + 2\operatorname{Re}(w\overline{z}) \leq |z|^2 + |w|^2 + 2|w\overline{z}| \\
&= |z|^2 + |w|^2 + 2|w||\overline{z}| = |z|^2 + |w|^2 + 2|w||z| = (|z| + |w|)^2.
\end{aligned}$$

Using induction, we get

$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|.$$

Homework. I, §1: 2 (c,f), 8, 10 (b,d,f,h).

1.2 Polar Form

For a complex number z, the expression $z = x + iy$ with $x, y \in \mathbb{R}$ is called the rectangular form of z. Now we introduce the polar form of a complex number. A point (x, y) that corresponds to $z = x + yi$ can be represented by polar coordinates (r, θ) such that $r \geq 0$, $\theta \in \mathbb{R}$, and

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Since $\cos^2\theta + \sin^2\theta = 1$, we see

$$r = \sqrt{x^2 + y^2} = |z|.$$

Now we find θ. The trivial case is $z = 0$, in which case $r = 0$ and θ can be any real number. Now we assume that $z \neq 0$. Then $r = |z| > 0$. The θ satisfies

$$\cos\theta = \frac{x}{|z|}, \qquad \sin\theta = \frac{y}{|z|}.$$

Such θ exists but is not unique because $\cos(x)$ and $\sin(x)$ both have period 2π. We call θ an argument of z and write $\arg z = \theta$. This expression is not accurate because if θ is an argument of z, then for every $n \in \mathbb{Z}$, $\theta + 2n\pi$ is also an argument of z. So $\arg z$ is a multivalued function. One may understand that $\arg z$ belongs to the quotient group $\mathbb{R}/(2\pi\mathbb{Z})$.

Examples. $\arg 1 = 0$, $\arg(-1) = \pi$, $\arg i = \frac{\pi}{2}$, $\arg(-i) = \frac{3\pi}{2}$, $\arg(-1 + i) = \frac{3\pi}{4}$.

Now we introduce the principal argument $\operatorname{Arg} z$ of a complex number $z \neq 0$. There are two different definitions in the literature:

1. $\operatorname{Arg} z$ is the unique argument of z that lies in the interval $[0, 2\pi)$.

2. $\operatorname{Arg} z$ is the unique argument of z that lies in the interval $(-\pi, \pi]$.

Lang's book uses the first definition. The second definition is also frequently used.

We use $e^{i\theta}$ as a shorthand for $\cos\theta + i\sin\theta$. The polar form of a complex number is

$$z = re^{i\theta}, \qquad r \geq 0, \quad \theta \in \mathbb{R}.$$

The polar form is useful because

1. Give a geometric explanation of complex multiplication.

2. Simplify the computations of powers and power roots.

3. Introduce the exponential function of complex numbers.

Theorem 1.2.1. *For $\theta_1, \theta_2 \in \mathbb{R}$,*

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

Proof.

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}. \end{aligned}$$

□

Thus, $(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = (r_1r_2)e^{i(\theta_1+\theta_2)}$. This means that $\arg(z_1z_2) = \arg z_1 + \arg z_2$. Since $\arg(1) = 0$, we have $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$. The addition and subtraction can be understood as the operation in the group $\mathbb{R}/(2\pi\mathbb{Z})$.

Now we consider the power function and power roots. Induction shows that $\arg(z^n) = n\arg z$ for any $n \in \mathbb{Z}$. This gives a method to quickly compute z^n in some cases because

$$z^n = |z|^n e^{in\arg z}.$$

We may also use the polar form to find the n-th root of a complex number z, i.e., $w \in \mathbb{C}$ such that $w^n = z$. If $z = 0$, then w has to be 0. Now suppose $z \neq 0$ and $\theta = \arg z$. First, we have $|w| = |z|^{1/n}$. Second, let θ be an argument of z. If ϕ is an argument of w, then $n\phi$ is an argument of z, which means that

$$n\phi = \theta + 2k\pi, \quad k \in \mathbb{Z}.$$

Thus, we get a sequence of roots:

$$w_k = |z|^{1/n}e^{i(\theta+2k\pi)/n}, \quad k \in \mathbb{Z}.$$

This does not mean that we have infinitely many n-th roots of z. In fact, it is clear that $w_{k+n} = w_k$. So there are totally n roots: $w_0, w_1, \ldots, w_{n-1}$. They lie on the circle centered at 0 with radius $|z|^{1/n}$, and the angle between any two of them is $2\pi/n$. So these roots are the vertices of a regular n-polygon.

At the end of this section, we introduce the exponential function:

$$\exp : \mathbb{C} \to \mathbb{C}$$

such that, for $z = x + yi \in \mathbb{C}$,

$$\exp(z) = e^x e^{yi} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y.$$

The $\exp(z)$ is also written as e^z. It has the following properties:

1. exp extends the real exponential function, i.e., if $z = x \in \mathbb{R}$, then the value of $\exp(z)$ agrees with that of $\exp(x)$ for real exponential function.

2. $e^{z_1}e^{z_2} = e^{z_1+z_2}$. This follows from the previous theorem, and extends the similar equality for real numbers.

3. $|e^z| = e^{\operatorname{Re} z} > 0$, which implies that $e^z \neq 0$.

4. $\operatorname{Im} z$ is an argument of e^z.

5. For any $w \in \mathbb{C} \setminus \{0\}$, there exists $z \in \mathbb{C}$ such that $w = e^z$. In fact, expressing $w = re^{i\theta}$ with $r > 0$, we may choose $z = \log r + i\theta$.

6. exp has period $2\pi i$, i.e., $\exp(z + 2n\pi i) = \exp(z)$ for any $n \in \mathbb{Z}$. This means that the z such that $e^z = w$ is not unique.

We now define the logarithm function $\log z$ for every $z \in \mathbb{C} \setminus \{0\}$ such that $\log z$ is the set of $w \in \mathbb{C}$ with $e^w = z$. From the above observation, we see that

$$\log z = \log |z| + i \arg z.$$

We may define the principal logarithm function by

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z.$$

The exact value depends on the choice of $\operatorname{Arg} z$. Then $\operatorname{Log} z \in \log z$, and so $e^{\operatorname{Log} z} = z$.

Example. The principal value of $\log(1 - i)$ has two possibilities:

$$\log(1 - i) = \log(\sqrt{2}) + i\frac{7}{4}\pi, \quad \log(1 - i) = \log(\sqrt{2}) - i\frac{1}{4}\pi,$$

depending on which definition we choose.

Homework. Chapter I, §1: 7; §2: 1 (a,c,d,h), 2 (b,c,f,g), 8, 11, 12
Additional problems

1. Compute the following principal logarithms using $\operatorname{Arg} z \in [0, 2\pi)$ and $\operatorname{Arg} z \in (-\pi, \pi]$, respectively:
$$\operatorname{Log}(-i), \quad \operatorname{Log}(\sqrt{3} + i), \quad \operatorname{Log}(-1 - i).$$

2. Find all $z \in \mathbb{C}$ in the rectangular form which solve $z^6 + 1 = 0$. The trigonometric functions must be evaluated.

3. Let $n \in \mathbb{N}$ with $n \geq 2$ and $z_0 \in \mathbb{C} \setminus \{0\}$. Let $w_1, \ldots, w_n$ be the distinct n-th roots of z_0. Prove that $\sum_{k=1}^{n} w_k = 0$.

4. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be distinct. State conditions in terms of computation of complex numbers, which make z_1, z_2, z_3, z_4 vertices of a square in the counterclockwise direction.

1.3 Complex Valued Functions

Let $S \subset \mathbb{C}$. A complex valued function defined on S is a function

$$f : S \to \mathbb{C}.$$

We have seen a number of complex valued functions such as $\operatorname{Re} z$, $\operatorname{Im} z$, $\overline{z}$, $|z|$, z^n, $n \in \mathbb{N}$, and $\exp(z)$, which are all defined on $\mathbb{C}$. The function z^{-1} and z^{-n}, $n \in \mathbb{N}$, are defined on $\mathbb{C} \setminus \{0\}$. A complex valued function can be written in the rectangular form:

$$f(x + yi) = u(x, y) + iv(x, y),$$

where u and v are real valued functions. For example,

1. $f(z) = \operatorname{Im} z$: $u(x, y) = y$, $v(x, y) = 0$.
2. $f(z) = \overline{z}$: $u(x, y) = x$, $v(x, y) = -y$.
3. $f(z) = z^2$: $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$.
4. $f(z) = e^z$: $u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$.
5. $f(z) = z^{-1}$: $u(x, y) = \frac{x}{x^2+y^2}$, $v(x, y) = -\frac{y}{x^2+y^2}$.
6. $f(z) = \operatorname{Log} z$: $u(x, y) = \log|x + iy| = \frac{1}{2}\log(x^2 + y^2)$ and $v(x, y) = \operatorname{Arg}(x + iy)$. Recall that there are two definitions of the principal argument. If $y > 0$, then in both definitions we have $v(x, y) = \operatorname{arccot}(x/y) = \frac{\pi}{2} - \arctan(x/y)$.

Now we define the complex hyperbolic functions. Recall that for $x \in \mathbb{R}$,

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}.$$

So we define for $z \in \mathbb{C}$,

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh x = \frac{e^z - e^{-z}}{2}, \quad \tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}.$$

Surprisingly, we will also use complex exponential functions to define complex trigonometric functions. First, we observe that, for $\theta \in \mathbb{R}$,

$$e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta;$$

$$e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta.$$

So $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ for $\theta \in \mathbb{R}$. This suggests us to define for $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}.$$

Most of the trigonometric identities still hold for these complex trigonometric functions. For example,

$$\cos^2 z + \sin^2 z = \frac{(e^{iz} + e^{-iz})^2}{4} + \frac{(e^{iz} - e^{-iz})^2}{-4} = \frac{e^{2iz} + 2 + e^{-2iz}}{4} - \frac{e^{2iz} - 2 + e^{-2iz}}{4} = 1.$$

But we may not use this equality to conclude that $|\cos z| \leq 1$ and $|\sin z| \leq 1$. In fact, cos and sin are unbounded functions.

Homework. I, §3: 4 (Add (c): The set of z with $|z| \geq 100$.); II, §3: 4.
Additional problems.

1. Express $\sin z$ and $\cos z$ in the rectangular form.

1.4 Topology of Complex Numbers

In this section, we review some topology concepts on $\mathbb{C}$.

The distance between $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is

$$d(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

One may easily check the triangle inequality:

$$d(z_1, z_2) + d(z_2, z_3) = |z_1 - z_2| + |z_2 - z_3| \geq |z_1 - z_3| = d(z_1, z_3).$$

So this distance is a metric space distance. The topology on $\mathbb{C}$ is generated by this distance. Note that the distance agrees with the Euclidean distance on $\mathbb{R}^2$. So the topology on $\mathbb{C}$ is the same as the topology on $\mathbb{R}^2$.

An open disc of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ is the set

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The closed disc is

$$\overline{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

In this course, a disc is always an open disc.

Let $U \subset \mathbb{C}$. We say that U is open if for every $\alpha \in U$, there is $r > 0$ such that

$$D(\alpha, r) \subset U.$$

In general, r depends on α. We have the following examples.

1. $\mathbb{C}$ is open because we may always choose $r = 1$.

2. The empty set $\emptyset$ is open because there is nothing to check.

3. The half plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is open. For $\alpha \in H$, we choose $r = \operatorname{Re} \alpha$. Note that, if $|z - \alpha| < r$, then $|\operatorname{Re} z - \operatorname{Re} \alpha| \le |z - \alpha| < r$, so $\operatorname{Re} z > \operatorname{Re} \alpha - r = 0$. Similarly, the half planes $\{\operatorname{Re} z < 0\}$, $\{\operatorname{Im} z > 0\}$, and $\{\operatorname{Im} z < 0\}$ are also open.

4. The open disc $D(z_0, R)$ is open. For $\alpha \in D(z_0, R)$, we choose $r = R - |\alpha - z_0|$. Note that, if $|z - \alpha| < r$, then $|z - z_0| \le |z_0 - \alpha| + |z - \alpha| < |\alpha - z_0| + r = R$, so $z \in D(z_0, R)$.

5. For any $R \ge 0$, the set $S = \{z \in \mathbb{C} : |z - z_0| > R\}$ is open. For $\alpha \in S$, we may choose $r = |\alpha - z_0| - R$.

The open sets satisfy the following properties:

1. If U_λ, $\lambda \in \Lambda$, is a family of open sets, then $\bigcup_{\lambda \in \Lambda} U_\lambda$ is also open.

2. If $U_1, \dots, U_n$ are open sets, then $\bigcap_{k=1}^n U_k$ is also open.

For example, the first quadrant $\{\operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ is the intersection of two half planes $\{\operatorname{Re} > 0\}$ and $\{\operatorname{Im} z > 0\}$, both of which are open, so it is open.

Let $S \subset \mathbb{C}$. A boundary point of S is a point α (may or may not lie in S) such that for any $r > 0$, $D(\alpha, r)$ intersects both S and $\mathbb{C} \setminus S$. We use ∂S to denote the set of boundary points of S. For example, $\partial\{\operatorname{Im} z > 0\} = \mathbb{R}$, $\partial D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$.

The closure of S, denoted by $\overline{S}$, is the union of S and ∂S. S is called closed if $\partial S \subset S$, i.e., $S = \overline{S}$. We have the following facts:

1. $z \in \overline{S}$ if and only if for every $r > 0$, $D(z, r) \cap S \ne \emptyset$.

2. $\overline{S}$ is closed for every $S \subset \mathbb{C}$.

3. S is closed if and only if $S^c := \mathbb{C} \setminus S$ is open.

4. If F_λ, $\lambda \in \Lambda$, is a family of closed sets, then $\bigcap_{\lambda \in \Lambda} F_\lambda$ is also closed.

5. If $F_1, \dots, F_n$ are closed sets, then $\bigcup_{k=1}^n F_k$ is also closed.

We see that the whole plane $\mathbb{C}$, empty set $\emptyset$, closed half plane $\{\operatorname{Re} z \ge 0\}$, and closed disc $\overline{D}(z_0, R)$ are closed because their complements are open. The same is true for a single point set $\{z_0\} = \overline{D}(z_0, 0)$. Using a finite union, we see that any finite set $\{z_1, \dots, z_n\}$ is closed.

We say that S is dense in $\mathbb{C}$ if its closure $\overline{S}$ equals to $\mathbb{C}$, i.e., for every $z \in \mathbb{C}$ and $r > 0$, $D(z, r) \cap S \ne \emptyset$.

If z_n, $n \in \mathbb{N}$, is a sequence of complex numbers, and $w \in \mathbb{C}$. We say that w is the limit of z_n or z_n tends to w, and write

$$w = \lim_{n \to \infty} z_n, \quad \text{or} \quad z_n \to w$$

if $|z_n - w| \to 0$, that is, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that if $n \ge N$, then $|z_n - w| < \varepsilon$. In this case, we say that (z_n) is a convergent sequence.

For a nonempty $S \subset \mathbb{C}$, every $w \in \overline{S}$ can be expressed as a limit of a sequence, which is contained in S, and vice versa. Thus, S is closed if and only if it contains the limits of all convergent sequences in S.

Since the topology on $\mathbb{C}$ is the same as the topology on $\mathbb{R}^n$, we have the following theorem.

Theorem 1.4.1. *$z_n \to w$ if and only if $\operatorname{Re} z_n \to \operatorname{Re} w$ and $\operatorname{Im} z_n \to \operatorname{Im} w$.*

Using the theorem, it is easy to check that $z_n \to z_0$ and $w_n \to w_0$ imply that $z_n \pm w_n \to z_0 \pm w_0$, $z_n w_n \to z_0 w_0$, and $z_n/w_n \to z_0/w_0$ if w_n and w_0 are not zero.

For example, if $z_n = x_n + iy_n$ and $w_n = u_n + iv_n$, then $z_n w_n = (x_n u_n - y_n v_n) + i(x_n v_n + y_n u_n)$. Since $x_n u_n - y_n v_n \to x_0 u_0 - y_0 v_0$ and $x_n v_n + y_n u_n \to x_0 v_0 + y_0 u_0$, we get $z_n w_n \to (x_0 u_0 - y_0 v_0) + i(x_0 v_0 + y_0 u_0) = z_0 w_0$.

A sequence of complex numbers (z_n) is said to be a Cauchy sequence if, given $\varepsilon > 0$, there exists N such that, if $m, n \geq N$, then $|z_m - z_n| < \varepsilon$. The triangle inequality implies that a convergent sequence is Cauchy. On the other hand, if (z_n) is a Cauchy sequence, then $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ are two real Cauchy sequences because, e.g., $|\operatorname{Re} z_m - \operatorname{Re} z_n| = |\operatorname{Re}(z_m - z_n)| \leq |z_m - z_n|$. From real analysis, they converge to two real numbers, say u and v. From the previous theorem, $w = u + vi$ is the limit of (z_n). Thus, a Cauchy sequence of complex numbers is convergent.

We say $S \subset \mathbb{C}$ is bounded if there is $R > 0$ such that $S \subset \overline{D}(0, R)$, i.e., $|z| \leq R$ for all $z \in S$.

Theorem 1.4.2. *Every convergent sequence is bounded.*

Proof. If (z_n) is convergent, then $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ are convergent sequences of real numbers, which have to be bounded. Since $|z_n| \leq |\operatorname{Re} z_n| + |\operatorname{Im} z_n|$, (z_n) is bounded as well. $\square$

The theorem below is a special case of a similar theorem for $\mathbb{R}^n$.

Theorem 1.4.3. **[Bolzano-Weierstrass]** *Every bounded sequence of complex numbers contains a convergent subsequence.*

Proof. Let (z_n) be a bounded sequence of complex numbers. Let $x_n = \operatorname{Re} z_n$ and $y_n = \operatorname{Im} z_n$, $n \in \mathbb{N}$. Since $|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z|$, (x_n) and (y_n) are two bounded sequences of real numbers. Apply the B-W theorem for real numbers to (x_n). We find a convergent subsequence (x_{n_k}) of (x_n). We now look at the subsequence (y_{n_k}), which may not be convergent, but is still bounded. Apply the theorem again to (y_{n_k}). We find a convergent subsequence $(y_{n_{k_l}})$ of (y_{n_k}). Then we come back to the x-sequence. As a subsequence of the convergent sequence (x_{n_k}), $(x_{n_{k_l}})$ is also convergent. Thus, $(z_{n_{k_l}})$ is a convergent subsequence of (z_n). $\square$

Now we recall the definition of compact sets. There are three equivalent definitions.

Definition 1.4.1. *A set $S \subset \mathbb{C}$ is said to be compact if the following is true. If $\{U_\alpha\ \alpha \in A\}$ is a family of open sets such that $S \subset \bigcup_{\alpha \in A} U_\alpha$, then there exists $\alpha_1, \ldots, \alpha_n \in A$ such that $S \subset \bigcup_{k=1}^{n} U_{\alpha_k}$.*

Definition 1.4.2. *A set $S \subset \mathbb{C}$ is said to be compact if every sequence in S contains a convergent subsequence, whose limit lies in S.*

Definition 1.4.3. *A set $S \subset \mathbb{C}$ is said to be compact if it is bounded and closed.*

The three definitions are equivalent for compact subsets of $\mathbb{C}$. The first definition works for all topology spaces, the second definition works for all metric spaces, and the third definition only works for the finite dimensional Euclidean spaces $\mathbb{R}^n$. Proving the equivalence between these definitions require some work. We omit the proof here.

Theorem 1.4.4. *Let $(S_n)_{n=1}^{\infty}$ be a sequence of nonempty compact subsets of $\mathbb{C}$ with $S_n \supset S_{n+1}$ for any $n \in \mathbb{N}$. Then the intersection of all S_n is not empty.*

Proof. Choose $z_n \in S_n$, $n \in \mathbb{N}$.Then (z_n) is a sequence in (S_1). Since S_1 is compact, (z_n) contains a convergent subsequence (z_{n_k}). Call the limit v. Then $v \in S_1$. Fix any $m \in \mathbb{N}$. If $k \geq m$, then $n_k \geq k \geq m$, so $z_{n_k} \in S_{n_k} \subset S_m$. Thus, the subsequence $(z_{n_k})_{k \geq m}$ is contained in S_m. Since v is also the limit of this sequence, and S_m is closed, we have $v \in S_m$. Since this is true for any $m \in \mathbb{N}$, we find that $v \in \bigcap_{m \in \mathbb{N}} S_m$. □

Let A and B be nonempty subsets of $\mathbb{C}$, the distance between A and B is defined to be

$$\operatorname{dist}(A, B) := \inf\{|z - w| : z \in A, w \in B\}.$$

Since $|z - w|$ is always nonnegative, $\operatorname{dist}(A, B) \geq 0$. If $A \cap B \neq \emptyset$, then $\operatorname{dist}(A, B) = 0$. But $\operatorname{dist}(A, B) = 0$ does not imply that $A \cap B \neq \emptyset$. For example, $A = \{\operatorname{Re} z > 0\}$ and $B = \{\operatorname{Re} z < 0\}$.

Theorem 1.4.5. *Let A and B be nonempty subsets of $\mathbb{C}$. Suppose A is compact and B is closed. Then there exist $z_0 \in A$ and $w_0 \in B$ such that $|z_0 - w_0| = \operatorname{dist}(A, B)$. In other words, the minimum of the set $\{|z - w| : z \in A, w \in B\}$ exists.*

Proof. From the definition of $\operatorname{dist}(A, B)$, we may find a sequence (z_n) in A and a sequence (w_n) in B such that $|z_n - w_n| \to \operatorname{dist}(A, B)$. Since A is compact, it is bounded. So (z_n) is a bounded sequence. Since $(|z_n - w_n|)$ is convergent, it is also a bounded sequence. From $|w_n| \leq |z_n| + |z_n - w_n|$ we see that (w_n) is also a bounded sequence. Since A is compact, (z_n) contains a convergent subsequence (z_{n_k}), whose limit, say $z_0 \in A$. Applying B-W Theorem to the bounded sequence (w_{n_k}), we find a convergent subsequence $(w_{n_{k_l}})$. Let w_0 be the limit. Since $(w_{n_{k_l}})$ is a sequence in B, and B is closed, we have $w_0 \in B$. Since $(z_{n_{k_l}})$ is a subsequence of (z_{n_k}), we know that $z_{n_{k_l}} \to z_0$. Finally, from $z_{n_{k_l}} \to z_0$ and $w_{n_{k_l}} \to w_0$ we claim that $|z_0 - w_0| = \lim |z_{n_{k_l}} - w_{n_{k_l}}| = \operatorname{dist}(A, B)$, which finishes the proof. To prove the claim, we use the triangle inequalities

$$|z_{n_{k_l}} - w_{n_{k_l}}| - |z_0 - w_0| \leq |z_{n_{k_l}} - z_0| + |w_0 - w_{n_{k_l}}|,$$

$$|z_0 - w_0| - |z_{n_{k_l}} - w_{n_{k_l}}| \leq |z_0 - z_{n_{k_l}}| + |w_{n_{k_l}} - w_0|,$$

which imply that $||z_{n_{k_l}} - w_{n_{k_l}}| - |z_0 - w_0|| \leq |z_{n_{k_l}} - z_0| + |w_{n_{k_l}} - w_0| \to 0$. □

The above theorem implies that if A is compact, B is closed, and $\operatorname{dist}(A, B) = 0$, then $A \cap B \neq \emptyset$. The theorem is not true if A and B are both closed. For example, $A = \mathbb{N}$ and $B = \{n + 1/n^2 : n \in \mathbb{N}\}$, $\operatorname{dist}(A, B) = 0$ but $A \cap B = \emptyset$.

Homework.

1. Prove that, for $z \in \mathbb{C}$, the sequence $(z^n)_{n=1}^{\infty}$ converges if and only if $|z| < 1$ or $z = 1$.

2. Let K be a nonempty compact subset of an open set $U \subset \mathbb{C}$. Show that there is $r > 0$ such that $D(z, r) \subset U$ for any $z \in K$. Note that the r does not depend on $z \in K$.

3. Let $S \subset \mathbb{C}$. We say that z_0 is an accumulation point of S if for every $r > 0$, the intersection $D(z_0, r) \cap S$ is an infinite set. Let $U \subset \mathbb{C}$ be an open set such that $S \subset U$. Suppose that S does not have any accumulation point contained in U. Prove that for any compact set $K \subset U$, the intersection $S \cap K$ is finite.

Let $S \subset \mathbb{C}$ and $\alpha \in \overline{S}$. Let $f : S \to \mathbb{C}$. We say that

$$w = \lim_{\substack{z \to \alpha \\ z \in S}} f(z).$$

if for every sequence z_n in S that converges to α, we have $f(z_n) \to w$. Equivalently, for every $\varepsilon > 0$, there is $\delta > 0$ such that if $z \in S$ and $|z - \alpha| < \delta$, then $|f(z) - w| < \varepsilon$. Here α may or may not lie in S. If $\alpha \in S$, then $f(\alpha)$ is defined. We say that f is continuous at α if

$$f(\alpha) = \lim_{\substack{z \to \alpha \\ z \in S}} f(z).$$

We say that f is continuous on S if it is continuous at every $\alpha \in S$. The following theorem obviously holds.

Theorem 1.4.6. *1. Let $S \subset \mathbb{C}$. Let $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ be continuous. Then $f + g$, $f - g$, and fg are continuous on S. If $g \neq 0$ on S, then f/g is continuous on S.*

2. Let $S, T \subset \mathbb{C}$. Let $f : S \to \mathbb{C}$ and $g : T \to \mathbb{C}$ be continuous such that $f(S) \subset T$. Then $g \circ f$ is continuous on S.

Here are some examples.

1. Let $C \in \mathbb{C}$. The function $f(z) = C$ for all $z \in \mathbb{C}$ is called constant. It is continuous because the if $z_n = C$ for all n, then $z_n \to C$. The function $f(z) = z$ for all $z \in \mathbb{C}$ is continuous because $z_n \to \alpha$ implies that $f(z_n) = z_n \to \alpha = f(\alpha)$.

2. Let $a_0, \ldots, a_n \in \mathbb{C}$. The function

$$P(z) = \sum_{k=0}^{n} a_k z^k = a_0 + a_1 z + \cdots a_n z^n$$

is called a complex polynomial, which is continuous from the above theorem. If all a_k are zero, then P is constant zero. Other wise, there is a biggest n_0 such that $a_{n_0} \neq 0$. Then we only need to sum up k from 0 to n_0. In this case, we say that the degree of P is n_0, and write $\deg(P) = n_0$.

3. $z \mapsto \operatorname{Re} z$ and $z \mapsto \operatorname{Im} z$ are continuous. For example, let $\alpha \in \mathbb{C}$ and fix $\varepsilon > 0$. Let $\delta = \varepsilon > 0$. If $|z - \alpha| < \delta$, then $|\operatorname{Re} z - \operatorname{Re} \alpha| = |\operatorname{Re}(z - \alpha)| \le |z - \alpha| < \delta = \varepsilon$.

Homework. Let $z_0 \in \mathbb{C}$ and $f(z) = |z - z_0|$. Show that f is continuous on $\mathbb{C}$.

Theorem 1.4.7. *Let $S \subset \mathbb{C}$ be a compact. Let $f : S \to \mathbb{C}$ be continuous. Then*

(i) $f(S) := \{f(z) : z \in S\}$ is also compact.

(ii) f is bounded on S, i.e., there exists $R < \infty$ such that $|f(z)| \le R$ for all $z \in S$.

Proof. (i) Let (w_n) be a sequence in $f(S)$. Then there is a sequence (z_n) in S such that $w_n = f(z_n)$, $n \in \mathbb{N}$. Since S is compact, (z_n) contains a convergent subsequence (z_{n_k}), whose limit, say z_0, lies in S. Since f is continuous at z_0, we have $w_{n_k} = f(z_{n_k}) \to f(z_0)$. Thus, (w_{n_k}) is a convergent subsequence of (w_n), whose limit is $f(z_0) \in f(S)$. This shows that $f(S)$ is compact.

(ii) This follows immediately from (i) since the compact set $f(S)$ is bounded. □

Definition 1.4.4. *Let $U \subset S \subset \mathbb{C}$. We say that U is relatively open in S if for every $z_0 \in U$, there is $r > 0$ such that*

$$D(z_0, r) \cap S \subset U.$$

Equivalently, U is relatively open in S if there is an open set $V \subset \mathbb{C}$ such that $U = V \cap S$. We say that $K \subset S$ is relatively closed in S if $S \setminus K$ is relatively open in S.

Note that, if S is open, then U is relatively open in S iff $U \subset S$ and U is open. We have the following theorem.

Theorem 1.4.8. *Let $S \subset \mathbb{C}$ and $f : S \to T \subset \mathbb{C}$. Then f is continuous iff for any relatively open set U in T, $f^{-1}(U) := \{z \in S : f(z) \in U\}$ is relatively open in S.*

Note that an open real interval is a relatively open subset of $\mathbb{R}$. So we get another way to show that $\{z : |z - z_0| < r\}$ and $\{z : |z - z_0| > r\}$ is open. This follows from that $f(z) = |z - z_0|$ is continuous, $\{z : |z - z_0| < r\} = f^{-1}((-\infty, r))$ and $\{z : |z - z_0| > r\} = f^{-1}((r, \infty))$. Similarly, to show that the half-plane $\{z : \operatorname{Im} z > 0\}$ is open, we may consider $f(z) = \operatorname{Im} z$.

Note that $\emptyset$ and S are relatively both open and closed in S.

Definition 1.4.5. *A set $S \subset \mathbb{C}$ is called connected if the only relatively open and closed sets in S are $\emptyset$ and S.*

Definition 1.4.6. *A set $S \subset \mathbb{C}$ is called path connected if for any $z_0, w_0 \in S$ there exists a continuous function $\gamma : [0, 1] \to S$ with $\gamma(0) = z_0$ and $\gamma(1) = w_0$.*

Remark. A path connected set must be connected. The converse may not be true. However, if S is open, then S is connected also implies that S is path connected.

Definition 1.4.7. *A nonempty connected open set $S \subset \mathbb{C}$ is called a (complex) domain.*

Examples of domains include discs, open half planes, and annulus $\{z \in \mathbb{C} : r < |z - z_0| < R\}$, where $z_0 \in \mathbb{C}$ and $R > r > 0$. If two discs are disjoint, then the union of them is not a domain.

1.5 Branch of the Complex Logarithm

Definition 1.5.1. *A branch of* $\log z$ *in an open set $U \subset \mathbb{C} \setminus \{0\}$ is a continuous function $L(z)$ defined on U such that $L(z) \in \log z$, i.e., $e^{L(z)} = z$ for every $z \in U$.*

Suppose $L(z) = u(z) + iv(z)$ is branch of $\log z$ in $U \subset \mathbb{C} \setminus \{0\}$. Then we have $u(z) = \log |z|$, which is continuous, and $v(z) \in \arg z$. This means that, finding a branch of $\log z$ is equivalent to finding a (continuous) branch of $\arg z$.

Now we consider the principal argument $\operatorname{Arg} z$ for $z \neq 0$. If $\operatorname{Arg} z \in [0, 2\pi)$, then Arg is continuous on $U_+ := \mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\}$. If $\operatorname{Arg} z \in (-\pi, \pi]$, then Arg is continuous on $U_- := \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$. So the principal arguments give branches of $\log z$ in U_+ and U_-. Similarly, we may define branches of $\log z$ in $\mathbb{C} \setminus \{xe^{i\theta_0} : x \geq 0\}$ for some fixed $\theta_0 \in \mathbb{R}$. Here we may choose $v(z)$ to lie in $[\theta_0, \theta_0 + 2\pi)$. The half line $\{xe^{i\theta_0} : x \geq 0\}$ is called a branch cut. We may also have a branch cut other than a half line. It is impossible to find a branch of $\log z$ in $\mathbb{C} \setminus \{0\}$.

Once a branch is fixed, it may be denoted "$\log z$" if no confusion can result. Then $\log z$ becomes a single valued function. Different branches can give different values for the logarithm of a particular complex number, however, so a branch must be fixed in advance in order for "$\log z$" to have a precise unambiguous meaning. For a fixed determination of the log on U and a fixed $\alpha \in \mathbb{C}$, we may define the complex power function

$$z^\alpha = \exp(\alpha \log z), \quad z \in U.$$

Homework III, §6: 1(d,e)&2 (d,e).

1.6 Complex Differentiability

Let U be an open set, and let $z_0 \in U$. Let $f : U \to \mathbb{C}$. We say that f is complex differentiable at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit is denoted by $f'(z_0)$ or $df/dz(z_0)$. The symbol $z \to z_0$ means that z stays in $U \setminus \{z_0\}$ and tends to z_0. Note that if $z = z_0$, the fractal has no meaning. An equivalent definition is

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

if the limit exists. If f is differentiable at every $z \in U$, then we say that f is differentiable on U. In this case, we also say that f is holomorphic on U. If $S \subset \mathbb{C}$ is not open, when we say that f is holomorphic on S, it means that there is an open set $U \supset S$ such that f is holomorphic on U. We say that f is an entire function if it is holomorphic on $\mathbb{C}$.

Examples.

1. Let $w_0 \in \mathbb{C}$. Let $f(z) = w_0$ for all $z \in \mathbb{C}$. Since $\frac{f(z)-f(z_0)}{z-z_0} = 0$ for any $z \neq z_0 \in \mathbb{C}$, we see that $f'(z) = 0$ for all $z \in \mathbb{C}$.

2. Let $f(z) = z$ for all $z \in \mathbb{C}$. Since $\frac{f(z)-f(z_0)}{z-z_0} = 1$ for any $z \neq z_0 \in \mathbb{C}$, we see that $f'(z) = 1$ for all $z \in \mathbb{C}$.

The basic properties for derivatives of real valued functions still hold here. The proofs are similar. Suppose f and g are both differentiable at z_0. Then

1. f and g are also continuous at z_0.

2. **Sum Rule.** $f + g$ is also differentiable at z_0 and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$.

3. **Product Rule.** fg is also differentiable at z_0 and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$. A special case is when g is constant C, and we get $(Cf)'(z_0) = Cf'(z_0)$.

4. **Quotient Rule.** If $g(z_0) \neq 0$, then f/g is differentiable at z_0, and

$$(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

   A special case is when f is constant 1, and we get $(\frac{1}{g})'(z_0) = \frac{-g'(z_0)}{g(z_0)^2}$.

In addition, we have the **Chain Rule**. If f is differentiable at z_0 and g is differentiable at $f(z_0)$, then $g \circ f$ is differentiable at z_0, and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Thus, if f and g are holomorphic on U, and $a \in \mathbb{C}$ then $f+g$, af, and fg are also holomorphic on U. If in addition, $g \neq 0$ on U, then f/g is holomorphic on U. If f is holomorphic on U, g is holomorphic on V, and $f(U) \subset V$, then $g \circ f$ is holomorphic on U.

If f is holomorphic on U, then f' is a well-defined function on U. If f' is also holomorphic on U, then we define $f'' = (f')'$. Similarly, we define $f''' = (f'')'$ if f'' is holomorphic on U. Another set of symbols is $f^{(0)} = f$ and $f^{(n+1)} = (f^{(n)})'$ if $f^{(n)}$ is well defined and is holomorphic on U.

Examples.

1. Using induction, we see that $\frac{d}{dz}z^n = nz^{n-1}$ for any $n \in \mathbb{N}$.

2. Every polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ is an entire function, and we have

$$P'(z) = \sum_{k=1}^{n} a_k k z^{k-1},$$

which is also a polynomial. If $\deg(P) = n \geq 1$, then $\deg(P') = n - 1$. If $\deg(P) = 0$, then $P' \equiv 0$. Thus, $P^{(n+1)} \equiv 0$ if $n = \deg(P)$.

3. Since $z^n \neq 0$ if $z \neq 0$, we see that $f(z) = z^{-n} = \frac{1}{z^n}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Using the quotient rule, we get

$$f'(z) = \frac{-nz^{n-1}}{(z^n)^2} = -nz^{-n-1}.$$

Thus, the formula $\frac{d}{dz} z^n = nz^{n-1}$ holds for any $n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$.

1.7 The Cauchy-Riemann Equations

Let U be an open set, and let $z_0 = x_0 + iy_0 \in U$. Let $f : U \to \mathbb{C}$. Write $f(x + iy) = u(x, y) + iv(x, y)$. It is not always efficient to use the definition and the derivative rules to determine whether f is complex differentiable at z_0. We now introduce a new method.

Theorem 1.7.1. *f is complex differentiable at z_0 if and only if the following two conditions hold.*

(i) Both u and v are totally differentiable at (x_0, y_0);

(ii) The partial derivatives of u and v satisfy the Cauchy-Riemann equation at (x_0, y_0), i.e.,

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

In addition, we have $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Recall that we say that u is totally differentiable at (x_0, y_0) if there exist $a, b \in \mathbb{R}$ such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{u(x, y) - u(x_0, y_0) - a(x - x_0) - b(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0,$$

where $(x, y) \to (x_0, y_0)$ means that $\sqrt{(x - x_0)^2 + (y - y_0)^2} \to 0$. The formula means that, near (x_0, y_0), $u(x, y)$ can be approximated by the function $h(x, y) = u(x_0, y_0) + a(x - x_0) + b(y - y_0)$. The graph of h is a plane that passes through the point $(x_0, y_0, u(x_0, y_0))$, which is a tangent plane of the graph of u. Setting $y = y_0$, we see that $\lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)-a(x-x_0)}{|x-x_0|} = 0$, which implies that $u_x(x_0, y_0) = a$. Similarly, $u_y(x_0, y_0) = b$. In general, the existence of the partial derivatives does not imply the totally differentiability. Recall the following proposition in multi-variable calculus, which can help us to check the totally differentiability.

Proposition 1.7.1. *Let U be an open set in $\mathbb{R}^2$, and u be a real valued function defined on U. Suppose u_x and u_y exist everywhere on U, and are continuous. Then u is totally differentiable everywhere on U.*

Proof of Theorem 1.7.1. First, we note that f is differentiable at z_0 and $w_0 = f'(z_0)$ if and only if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0) - w_0(z - z_0)}{z - z_0} = 0.$$

Since $z_n \to 0$ if and only if $|z_n| \to 0$, the above formula is equivalent to

$$\lim_{z \to z_0} \frac{f(z) - f(z_0) - w_0(z - z_0)}{|z - z_0|} = 0.$$

Writing $z = x + yi$ and $w_0 = a + bi$, we find that $z \to z_0$ is the same as $(x, y) \to (x_0, y_0)$, and

$$\frac{f(z) - f(z_0) - w_0(z - z_0)}{|z - z_0|} = \frac{u(x, y) - u(x_0, y_0) - a(x - x_0) + b(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

$$+i \frac{v(x, y) - v(x_0, y_0) - b(x - x_0) - a(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}.$$

Thus, $f'(z_0) = a + bi$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{u(x, y) - u(x_0, y_0) - a(x - x_0) + b(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{v(x, y) - v(x_0, y_0) - b(x - x_0) - a(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0,$$

which is equivalent to that u and v are totally differentiable at (x_0, y_0), and $u_x(x_0, y_0) = v_y(x_0, y_0) = a$ and $-u_y(x_0, y_0) = v_x(x_0, y_0) = b$. This finishes the proof. $\square$

Corollary 1.7.1. *Suppose u_x, u_y, v_x, v_y are all continuous on U, and the Cauchy-Riemann equation $u_x = v_y$ and $u_y = -v_x$ holds throughout U. Then f is holomorphic on U, and $f' = u_x + iv_x$.*

Examples.

1. If f is an entire function ($U = \mathbb{C}$) that satisfies $f' \equiv 0$, then we can conclude that f is constant. The reason is: from $f' = u_x + iv_x = v_y - iu_y$ we see that $u_x = u_y = v_x = v_y \equiv 0$. From Real Analysis, we see that both u and v are constant. So f is constant. If $U \neq \mathbb{C}$, whether $f' \equiv 0$ implies that f is constant depends on the connectedness of U.

2. Recall the exponential function $\exp(x+yi) = e^x(\cos y + i\sin y)$ defined on $\mathbb{C}$. We have $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Computing their partial derivatives, we get

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y, \quad e^x \cos y.$$

All of these functions are continuous on $\mathbb{R}^2$. Thus, u and v are totally differentiable at every $(x,y) \in \mathbb{R}^2$ Since u and v satisfy the C-R equation everywhere, exp is holomorphic on $\mathbb{C}$. Moreover, we have

$$\exp'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = \exp(z).$$

3. Since $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, they are both holomorphic on $\mathbb{C}$, and

$$\cos' z = \frac{ie^{iz} - ie^{-iz}}{2} = -\sin z, \quad \sin' z = \frac{ie^{iz} - (-i)e^{-iz}}{2i} = \cos z.$$

Similarly, $\cosh' z = \sinh z$ and $\sinh' z = \cosh z$.

4. Since $\tan z = \frac{\sin z}{\cos z}$ and $\cot z = \frac{\cos z}{\sin z}$, $\tan z$ is holomorphic on $\mathbb{C} \setminus \{n\pi + 1/2\pi : n \in \mathbb{Z}\}$, and $\cot z$ is holomorphic on $\mathbb{C} \setminus \{n\pi : n \in \mathbb{Z}\}$. From the quotient rule,

$$\tan' z = \frac{\sin' z \cos z - \sin z \cos' z}{\cos^2 z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z};$$

$$\cot' z = \frac{\cos' z \sin z - \cos z \sin' z}{\sin^2 z} = \frac{-\sin^2 z - \cos^2 z}{\sin^2 z} = -\frac{1}{\sin^2 z}.$$

5. We will prove later that any branch of $\log z$ is holomorphic. In the homework, we will consider the principal logarithm.

Homework.

1. Let $\operatorname{Log} z$ be the principal logarithm function defined using $\operatorname{Arg} z \in (-\pi, \pi]$. Prove that Log is holomorphic on $\mathbb{C} \setminus \{x \in \mathbb{R} : x \le 0\}$, and $\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$. Hint: Apply the C-R equations to $U_1 = \{\operatorname{Im} z > 0\}$, $U_2 = \{\operatorname{Re} z > 0\}$, and $U_3 = \{\operatorname{Im} z < 0\}$ separately.

2. (i) Let f and g be two entire functions such that $f'(z) = g'(z)$ for all $z \in \mathbb{C}$. Show that there is a constant $C \in \mathbb{C}$ such that $f = g + C$. (ii) Let f be an entire function and $n \in \mathbb{N}$. Suppose that $f^{(n)} \equiv 0$. Show that f is a polynomial of degree no more than $n-1$.

3. Let f be an entire function. Suppose that $|f|$ is constant. Prove that f is constant. Hint: $f \equiv C$ implies that $u^2 + v^2 \equiv C^2$. Take partial derivatives and apply the C-R equations.

4. Let f be an entire function. Suppose that $f'(z) = f(z)$ for all $z \in \mathbb{C}$. Prove that there is a constant $C \in \mathbb{C}$ such that $f(z) = Ce^z$.

Chapter 2

Power Series

Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers, we will study the following power series

$$\sum_{n=0}^{\infty} a_n z^n := \lim_{n\to\infty} \sum_{k=0}^{n} a_k z^k = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots .$$

Recall that $0^0 = 1$. Since $0^n = 0$ for $n \geq 1$, the series always converge at 0, and the sum is a_0.

We are interested in two questions:

1. For what $z \in \mathbb{C}$ does the series converge/diverge?
2. If we define $f(z) = \sum_{n=0}^{\infty} a_n z^n$, what property does f have?

The following two theorems answer these questions.

Theorem 2.0.2. [Main Theorem 1] *Let*

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}. \tag{2.1}$$

Then the series converges if $|z| < R$, and diverges if $|z| > R$.

Such R is called the radius of convergence, or simply radius. Since lim sup takes value in $[0, \infty]$, $R \in [0, \infty]$ as well. Here we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. If $R = 0$, the series converges only at $z = 0$. If $R = \infty$, the series converges for all $z \in \mathbb{C}$. If $R \in (0, \infty)$, the series may converge or diverge when $|z| = R$.

We may use (2.1) to calculate the radius of convergence. Here are some tests, which work for particular cases.

Root Test. If $\lim |a_n|^{1/n}$ exists, then $\limsup_{n\to\infty} |a_n|^{1/n}$ equals to this limit, so R is the reciprocal of the limit.

Ratio Test. If $\lim \frac{|a_n|}{|a_{n+1}|}$ exists, then R equals the limit.

The ratio test is valid because for a sequence of positive numbers (r_n), if $\lim r_{n+1}/r_n$ exists, then $\lim r_n^{1/n}$ also exists, and the two limits are equal. This is a result in real analysis.

In the following examples, we will compute the radius of convergence of some power series. Recall that $R = 1/\lim |a_n|^{1/n}$ or $R = \lim |a_n|/|a_{n+1}|$ if either limit exists.

Examples.

1. The radius of $\sum z^n$ is 1 because $1^{1/n} \to 1$. From a homework problem, the limit is $\frac{1}{1-z}$.
2. The radius of $\sum n! z^n$ is 0 because $n!/(n+1)! = 1/(n+1) \to 0$.
3. The radius of $\sum \frac{z^n}{n!}$ is ∞ because $\frac{1}{n!}/\frac{1}{(n+1)!} = n+1 \to \infty$.
4. The radius of $\sum \frac{n!}{n^n} z^n$ is e because $\frac{n!}{n^n}/\frac{(n+1)!}{(n+1)^{n+1}} = (\frac{n+1}{n})^n \to e$.
5. For $\alpha \in \mathbb{C}$, consider the binomial series $\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$, where $\binom{\alpha}{0} = 1$ and
$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad n \geq 1.$$
The radius is ∞ if $\alpha \in \mathbb{Z}$ and $\alpha \geq 0$ because in this case $\binom{\alpha}{n} = 0$ for $n \geq \alpha + 1$. If $\alpha \notin \{0, 1, 2, \ldots\}$, then $\binom{\alpha}{n}$ never vanishes, and $|\binom{\alpha}{n}/\binom{\alpha}{n+1}| = |\frac{n+1}{\alpha-n}| \to 1$, which implies that $R = 1$.

Remark. The binomial series in the last example converges to the complex power function $(1+z)^\alpha$, where the branch of $\log(1+z)$ in $U = \{|z| < 1\}$ is chosen such that $\log(1) = 0$.

Homework. II, §2: 4 (a,c,f,g), 10.
Additional Problem:
Let $\sum a_n z^n$ be a power series with radius R. Answer the following questions with explanation.

(i) If the series diverges at $z = 3 - 4i$, what can you say about R?

(ii) If the series converges for every $z \in D(0,1)$, what can you say about R?

(iii) What is the radius of $\sum a_n z^{2n}$?

Theorem 2.0.3. [Main Theorem 2] *Suppose $R \in (0, \infty]$. Then $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on $D(0,R)$. Moreover, $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, whose radius is also R.*

Note that since f' is also the sum of a power series in $D(0,R)$, it is also holomorphic and f'' is still the sum of a power series. Repeating this argument, we find that f is infinitely many times complex differentiable.

One important example is the power series
$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We have seen that $R = \infty$. Applying Theorem 2.0.3, we see that $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is an entire function, and

$$f'(z) = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = f(z).$$

From a homework problem, we see that $f(z) = Ce^z$ for some $C \in \mathbb{C}$. Since $f(0) = a_0 = 1 = e^0$, we see $C = 1$. Thus, the function e^z has a power series expansion:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots, \quad z \in \mathbb{C}.$$

For any $r \in \mathbb{C}$, the function e^{rz} also has a power series expansion:

$$e^{rz} = \sum_{n=0}^{\infty} \frac{(rz)^n}{n!} = \sum_{n=0}^{\infty} \frac{r^n}{n!} z^n, \quad z \in \mathbb{C}.$$

From the definition of $\cos z$ an $\sin z$, we get

$$\cos z = \frac{1}{2}\Big(\sum_{n=0}^{\infty} \frac{i^n}{n!} z^n + \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} z^n\Big) = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} \cdot \frac{i^n}{n!} z^n = \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!} z^{2k}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots, \quad z \in \mathbb{C}.$$

The third "=" holds because $(1 + (-1)^n)/2 = 0$ when n is odd, $= 1$ when n is even, and we change the index using $n = 2k$. Since the series converges for every $z \in \mathbb{C}$, its radius of convergence is ∞. In this case, we can not use the ratio test because $a_n = 0$ if n is odd. Similarly,

$$\sin z = \frac{1}{2i}\Big(\sum_{n=0}^{\infty} \frac{i^n}{n!} z^n - \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} z^n\Big) = \sum_{n=0}^{\infty} \frac{1-(-1)^n}{2} \cdot \frac{i^{n-1}}{n!} z^n = \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k+1)!} z^{2k+1}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{7!} + \cdots, \quad z \in \mathbb{C}.$$

Here we change the index using $n = 2k+1$. The radius of convergence of this power series is also ∞, and $a_n = 0$ if n is even.

2.1 Series of Complex Numbers

Let (z_n) be a sequence of complex numbers. Consider the series

$$\sum_{n=1}^{\infty} z_n.$$

We define the partial sum

$$s_n = \sum_{k=1}^{n} z_k = z_1 + \cdots + z_n.$$

We say that the series converges if there is $w \in \mathbb{C}$ such that

$$\lim_{n\to\infty} s_n = w,$$

in which case we say that w is equal to the sum of the series, that is

$$w = \sum_{n=1}^{\infty} z_n.$$

If (s_n) diverges, we say that the series diverges.

If $\sum \alpha_n$ and $\sum \beta_n$ are two convergent series, and $C \in \mathbb{C}$, then $\sum C\alpha_n$ and $\sum(\alpha_n + \beta_n)$ also converge, and

$$\sum C\alpha_n = C \sum \alpha_n;$$

$$\sum(\alpha_n + \beta_n) = \sum \alpha_n + \sum \beta_n.$$

If a series $\sum z_n$ converges, then $z_n \to 0$. In fact, let (s_n) be the partial sum sequence. Then (s_n) and (s_{n-1}) both converge to the same limit, which implies that $z_n = s_n - s_{n-1}$ converges to 0. This means that, if (z_n) diverges or does not tend to 0, then $\sum z_n$ diverges.

Let $\sum \alpha_n$ be a series of complex numbers. We say that this series converges absolutely if the non-negative series

$$\sum |\alpha_n|$$

converges. We claim that if a series converges absolutely, then it converges in the usual sense.

Indeed, let $s_n = \sum_{k=1}^{n} \alpha_k$ and $t_n = \sum_{k=1}^{n} |\alpha_k|$, $n \in \mathbb{N}$. For $m \leq n$ we have

$$s_n - s_m = \alpha_{m+1} + \cdots + \alpha_n.$$

Hence

$$|s_n - s_m| \leq |\alpha_{m+1}| + \cdots + |\alpha_n| = t_n - t_m.$$

Assuming the absolute convergence, (t_n) is a Cauchy sequence, which implies that (s_n) is also a Cauchy sequence from the above inequality. Thus, (s_n) converges.

From calculus, we have the comparison test for convergence.

Comparison Test. Let $\sum \alpha_n$ be a series of complex numbers. Let $\sum c_n$ be a convergent series of nonnegative real numbers. If $|\alpha_n| \leq c_n$ for all n, then the series $\sum \alpha_n$ converges absolutely.

2.2 Sequence and Series of functions

Let $S \subset \mathbb{C}$. Let $f_n : S \to \mathbb{C}$, $n \in \mathbb{N}$. We say that the sequence (f_n) converges pointwise on S if for every $z \in S$, the sequence of numbers $(f_n(z))$ converges. We now define uniformly convergence. For $f : S \to \mathbb{C}$, the Sup norm of f on S is defined to be

$$\|f\|_S = \|f\| = \sup_{z \in S} |f(z)|.$$

We see that $\|f\| \geq 0$; $\|f\| = 0$ if and only if f is constant 0; for any $C \in \mathbb{C}$, $\|Cf\| = |C|\|f\|$; and the following triangle inequality holds:

$$\|f + g\| \leq \|f\| + \|g\|.$$

To prove the triangle inequality, note that for every $z \in S$,

$$|f(z) + g(z)| \leq |f(z)| + |g(z)| \leq \|f\| + \|g\|,$$

and then take the supremum over $z \in S$.

We say that (f_n) converges uniformly on S if there is $f : S \to \mathbb{C}$ such that

$$\lim_{n\to\infty} \|f_n - f\|_S = 0.$$

An equivalent definition is: for every $\varepsilon > 0$, there is $N \in \mathbb{N}$, such that if $n > N$, then $\|f_n - f\|_S < \varepsilon$. Since for each $z \in S$, $|f_n(z) - f(z)| \leq \|f_n - f\|_S$, we see that the uniformly convergence implies the pointwise convergence.

Theorem 2.2.1. *Let (f_n) be a sequence of continuous functions on S, which converges uniformly to f on S. Then f is also continuous on S.*

Proof. We will use the so-called $\varepsilon/3$-argument. Let $z_0 \in S$. Let $\varepsilon > 0$. Since $\|f_n - f\|_S \to 0$, there is $N \in \mathbb{N}$ such that $\|f_N - f\|_S < \varepsilon/3$. Since f_N is continuous on S, there is $\delta > 0$ such that if $z \in S$ and $|z - z_0| < \delta$, then $|f_N(z) - f_N(z_0)| < \varepsilon/3$, which then implies that

$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

So f is continuous at z_0. This shows that f is continuous on S. □

We say that (f_n) is a uniformly Cauchy sequence, if given $\varepsilon > 0$ there exists N such that if $m, n \geq N$, then $\|f_n - f_m\| < \varepsilon$.

Theorem 2.2.2. *A sequence of functions (f_n) on S is uniformly Cauchy on S if and only if it converges uniformly on S.*

Proof. First, suppose $f_n \to f$ uniformly on S. Let $\varepsilon > 0$. There is N such that $n \geq N$ implies that $\|f_n - f\|_S < \varepsilon/2$. If $n, m \geq N$, then

$$\|f_n - f_m\| \leq \|f_n - f\| + \|f_m - f\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, (f_n) is uniformly Cauchy on S.

Now suppose that (f_n) is uniformly Cauchy on S. Then it converges pointwise on S because for each $z \in S$, $|f_n(z) - f_m(z)| \leq \|f_n - f_m\| < \varepsilon$ if $m, n \geq N$, which implies that $(f_n(z))$ is a Cauchy sequence of complex numbers. Let $f : S \to \mathbb{C}$ be the pointwise limit of (f_n). There exists N such that if $m, n \geq N$, then $\|f_n - f_m\| < \varepsilon/2$, which implies that $|f_n(z) - f_m(z)| < \varepsilon/2$ for all $z \in S$. From triangle inequality, we get

$$|f_n(z) - f(z)| \leq |f_n(z) - f_m(z)| + |f_m(z) - f(z)| < \varepsilon/2 + |f_m(z) - f(z)|.$$

Fix $n \geq N$ and let $m \to \infty$. Since $f_m(z) \to f(z)$, we get $|f_n(z) - f(z)| \leq \varepsilon/2 < \varepsilon$. This holds for any $z \in S$ and $n \geq N$. So (f_n) converges to f uniformly on S. □

Consider a series of functions, $\sum f_n$, where each f_n is defined on S. Let $s_n = \sum_{k=1}^{n} f_k : S \to \mathbb{C}$ be the partial sum. We say that the series converges pointwise/uniformly if the sequence of functions (s_n) converges pointwise/uniformly. A series $\sum f_n$ is said to converge absolutely if the series $\sum |f_n|$ converges pointwise. For example, if $\sum f_n$ converges uniformly to f on S, and each f_n is continuous on f, then the partial sums s_n are all continuous, so f is also continuous.

We have the following comparison test.

Theorem 2.2.3. *Let (c_n) be a sequence of nonnegative real numbers, and assume that $\sum c_n$ converges. Let (f_n) be a sequence of functions on S such that $\|f_n\| \leq c_n$ for all n. Then $\sum f_n$ converges uniformly and absolutely.*

Proof. To prove that $\sum f_n$ converges absolutely, note that for each $z \in S$, $|f_n(z)| \leq \|f_n\| \leq c_n$. So we can apply the comparison test for the series of complex numbers. To prove that $\sum f_n$ converges uniformly, it suffices to show that the partial sum sequence $s_n = \sum_{k=1}^{n} f_k$, $n \in \mathbb{N}$, is a uniformly Cauchy sequence. In fact, for $n \geq m$,

$$\|s_n - s_m\| = \| \sum_{k=m+1}^{n} f_k\| \leq \sum_{k=m+1}^{n} \|f_k\| \leq \sum_{k=n+1}^{m} c_k,$$

which tends to 0 as $n, m \to \infty$. Thus, (s_n) is a uniformly Cauchy sequence. □

2.3 Radius of Power series

Now we come back to the power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $(a_n)_{n=0}^{\infty}$ is a sequence of complex numbers. Before proving Main Theorem 1, let's recall the definition of $\limsup$.

The limsup of a sequence of real numbers (x_n) is defined to be the infimum of the following decreasing sequence:

$$U_n = \sup\{x_k : k \geq n\}, \quad n \in \mathbb{N}.$$

If $L > \limsup x_n$, then there is some U_n such that $L > U_n$, which implies that $x_k < L$ for $k \geq n$. If $L < \limsup x_n$, then $U_n > L$ for each n, which implies that for every $n \in \mathbb{N}$ there is $k \geq n$ such that $x_k > L$.

We are now ready to prove Main Theorem 1. In fact, we can prove a little bit more.

Theorem 2.3.1. *Consider the power series $\sum a_n z^n$. Suppose $\frac{1}{R} = \limsup |a_n|^{1/n}$. Then*

(i) The series diverges for $|z| > R$.

(ii) The series converges absolutely for $|z| < R$.

(iii) If $R > r > 0$, then the series converges uniformly on $\{|z| \leq r\}$.

Proof. (i) Suppose $|z| > R$. Then $1/|z| < 1/R = \limsup |a_n|^{1/n}$. Thus, for every $n \in \mathbb{N}$ there is $k \geq n$ such that $|a_k|^{1/k} > 1/|z|$, which then implies that $|a_k z^k| > 1$. Thus the sequence $(a_n z^n)$ does not converge to 0, and the series $\sum a_n z^n$ must diverge. This finishes the proof of (i). If $R = 0$ then (ii) and (iii) must hold because there is nothing to check. Suppose $R > 0$. Let $r \in (0, R)$, and let $L \in (r, R)$. Since $L < R$, $1/L > 1/R = \limsup |a_n|^{1/n}$. So there is $N \in \mathbb{N}$ such that for $n \geq N$, $|a_n|^{1/n} < 1/L$, which implies that $|a_n L^n| < 1$. Thus, the sequence $(a_n L^n)_{n=0}^{\infty}$ is bounded. Suppose for some $C < \infty$, $|a_n L^n| \leq C$ for all n. Let $x = r/L \in (0, 1)$. Then

$$|a_n| r^n = |a_n L^n| x^n \leq C x^n, \quad n \geq 0.$$

From real analysis, $\sum x^n$ converges. Thus, $\sum |a_n| r^n$ converges from the comparison test. Since for $|z| \leq r$, $|a_n z^n| \leq |a_n| r^n$, From the previous theorem, $\sum a_n z^n$ converges absolutely and uniformly on $\{|z| \leq r\}$. This finishes the proof of (iii). Finally, since $\sum |a_n z^n|$ converges pointwise on $\{|z| \leq r\}$ for every $r \in (0, R)$, it must converge pointwise on $\{|z| < R\}$. This finishes the proof of (ii). $\square$

If $R > 0$, and we define $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then from (iii) we conclude that f is continuous on $\{|z| \leq r\}$ for any $0 < r < R$, which then implies that f is continuous on $\{|z| < R\}$. In general, the series may or may not converge uniformly on $\{|z| < R\}$.

Homework.

1. Prove that the series $\sum_{n=0}^{\infty} z^n$ does not converge uniformly on $D(0, 1)$.

2.4 Differentiation of Power Series

We now come to the proof of Main Theorem 2. Although we know that $\frac{d}{dz}z^0 = 0$ and $\frac{d}{dz}z^n = nz^{n-1}$ for $n \geq 1$, we can not conclude immediately that $\frac{d}{dz}\sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Lemma 2.4.1. *The series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.*

Proof. Since $\sum_{n=1}^{\infty} n a_n z^{n-1}$ and $\sum_{n=1}^{\infty} n a_n z^n$ converge for the same z, they have the same radius of convergence. Since $\lim n^{1/n} = 1 \in (0, \infty)$, we get

$$\limsup |na_n|^{1/n} = \lim n^{1/n} \limsup |a_n|^{1/n} = \limsup |a_n|^{1/n}.$$

So $\sum_{n=1}^{\infty} n a_n z^n$ and $\sum_{n=0}^{\infty} a_n z^n$ have the same radius of convergence. $\square$

Proof of Main Theorem 2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ on $D(0,R)$. We will show that $f'(z_0) = g(z_0)$ for every $z_0 \in D(0,R)$. Fix $z_0 \in D(0,R)$. We need to show that $\frac{f(z)-f(z_0)}{z-z_0} - g(z_0) \to 0$ as $z \to z_0$. We find that

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = \sum_{n=1}^{\infty} a_n \Big(\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \Big)$$

$$= \sum_{n=1}^{\infty} a_n (z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1} - n z_0^{n-1})$$

$$= \sum_{n=2}^{\infty} a_n \sum_{k=1}^{n-1} (z^k - z_0^k) z_0^{n-1-k} = (z - z_0) \sum_{n=2}^{\infty} a_n \sum_{k=1}^{n-1} z_0^{n-1-k} \sum_{j=0}^{k-1} z^j z_0^{k-1-j}$$

$$= (z - z_0) \sum_{n=2}^{\infty} a_n \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} z^j z_0^{n-2-j}.$$

Choose $L \in (|z_0|, R)$. Let $\varepsilon = L - |z_0|$. If $|z - z_0| < \varepsilon$, then $|z|, |z_0| < L$, and so

$$\Big| \sum_{n=2}^{\infty} a_n \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} z^j z_0^{n-2-j} \Big| \leq \sum_{n=2}^{\infty} |a_n| \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} L^{n-2}$$

$$= \sum_{n=2}^{\infty} |a_n| \frac{(n-1)(n-2)}{2} L^{n-2}.$$

Applying the previous lemma twice, we find that $\sum_{n=2} a_n (n-1)(n-2) z^{n-2}$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n z^n$. Since $L \in (0, R)$, the series $\sum_{n=2}^{\infty} a_n (n-1)(n-2) L^{n-2}$ converges absolutely. Thus, $C := \sum_{n=2}^{\infty} |a_n| \frac{(n-1)(n-2)}{2} L^{n-2} < \infty$. So we get the inequality

$$\Big| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \Big| \leq C |z - z_0|, \quad |z - z_0| < \varepsilon/2.$$

This shows that $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} - g(z_0) = 0$ as desired. $\square$

We now see that, if f is the sum of a power series with radius $R > 0$, then f is holomorphic, and f' is the sum of another power series, which also have radius R. We may further differentiate f'. So we see that f is infinitely many times complex differentiable.

On the other hand, the Main Theorem 2 also gives a method to find a holomorphic function F on $D(0, R)$, whose derivative is $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Such F is expressed by a power series: $F(z) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$, which also have radius R.

Differentiating the power series m times, we get

$$f^{(m)}(z) = \Big(\frac{d}{dz}\Big)^m \sum_{n=0}^{\infty} a_n z^n = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1) a_n z^{n-m}.$$

Since the value of $f^{(m)}(0)$ is the coefficients of the constant term, we get $f^{(m)}(0) = m! a_m$. Thus,

$$a_n = \frac{f^{(n)}(0)}{n!}, \quad n = 0, 1, 2, \ldots.$$

This means that we can recover the coefficients a_n from the n-th derivative of f at 0.

We will often consider series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

We call it a power series centered at z_0. If $\frac{1}{R} = \limsup |a_n|^{1/n}$, then the series converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. If $R > 0$, then the series converges to a holomorphic function f on $D(z_0, R)$, which is infinity times complex differentiable, and we have

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

This means that we may rewrite the series as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

Homework. II, §5: 1, 4, 6(a).

2.5 Analytic Functions and Uniqueness Theorem

Definition 2.5.1. *Let U be an open set and $f : U \to \mathbb{C}$. We say that f is analytic on U if for every $z_0 \in U$, there is $r > 0$ and a sequence of complex numbers $(a_n)_{n=0}^{\infty}$ such that $D(z_0, r) \subset U$ and $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ on $D(z_0, r)$.*

From the differentiability of power series we see that, if f is analytic on U, then it is holomorphic on U, and its derivative is still analytic on U. So such f is infinitely many times complex differentiable on U. Moreover, at every $z_0 \in U$, the coefficients a_n of the power series expansion of f centered at z_0 can be calculated using $a_n = \frac{f^{(n)}(z_0)}{n!}$, $n = 0, 1, 2, \cdots$.

In the next chapter, we will show that a holomorphic function is also analytic. Thus, if a function is complex differentiable on an open set, then it is infinitely many times complex differentiable. There is no such phenomena in real analysis.

Let f be analytic in U. A zero of f is some $z \in U$ such that $f(z) = 0$. Fix $z_0 \in U$. Let a_n, $n \geq 0$, be the coefficients of the power series expansion of f at z_0. This means that for some $r > 0$, $D(z_0, r) \subset U$ and $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ on $D(z_0, r)$. There are three cases about the behavior of the zeros of f near z_0.

1. If $a_0 \neq 0$, then $f(z_0) \neq 0$. Since f is continuous, there is $r_1 \in (0, r)$ such that $D(z_0, r_1)$ contains no zero of f.

2. If $a_0 = 0$ but not all a_n's are 0, then there is some smallest $m \in \mathbb{N}$ such that $a_m \neq 0$. Then there is $r > 0$ such that $f(z) = (z - z_0)^m g(z)$ and $g(z) = \sum_{k=0}^{\infty} a_{k+m}(z - z_0)^k$ on $D(z_0, r)$. Since $g(z_0) = a_m \neq 0$, there is $r_1 \in (0, r)$ such that $D(z_0, r_1)$ contains no zero of g. Since $(z - z_0)^m = 0$ if and only if $z = z_0$, we find that $D(z_0, r_1)$ contains only one zero of f, which is z_0.

3. If all a_n's are 0, then f is constant 0 on $D(z_0, r)$. In this case, every $z \in D(z_0, r)$ is a zero of f.

Let $S \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$. Recall that z_0 is an accumulation point of S if for every $r > 0$, $D(z_0, r)$ contains infinitely many elements in S. Note that this implies that z_0 lies in the closure of S: $\overline{S}$. However, $z_0 \in \overline{S}$ may not be an accumulation point of S. For example, any finite set is closed, but has no accumulation point; the set $\{1/n : n \in \mathbb{N}\}$ has only one accumulation point, which is 0.

Lemma 2.5.1. *z_0 is an accumulation point of S if and only if there is a sequence (z_n) in $S \setminus \{z_0\}$ such that $z_n \to z_0$.*

Proof. If z_0 is an accumulation point of S, then for any $n \in \mathbb{N}$, there is $z_n \in S \cap (D(z_0, \frac{1}{n}) \setminus \{z_0\})$. Then (z_n) is a sequence in $S \setminus \{z_0\}$. From $|z_n - z_0| < \frac{1}{n}$ we see that $z_n \to z_0$. This proves the "only if" part. On the other hand, suppose (z_n) is a sequence in $S \setminus \{z_0\}$, which tends to z_0. If z_0 is not an accumulation point of S, then there exists $r > 0$ such that $D(z_0, r) \cap S$ is finite, which then implies that $D(z_0, r) \cap \{z_n : n \in \mathbb{N}\}$ is finite. This shows that $\inf_{n \in \mathbb{N}} |z_n - z_0| > 0$, which contradicts that $z_n \to z_0$. This proves the "if" part. $\square$

Theorem 2.5.1 (Uniqueness Theorem). *Let U be a domain.*

(i) Suppose f is analytic in U, and is not constant 0. Then the set of zeros of f has no accumulation point in U.

(ii) *Suppose both f and g are analytic on U, and there is $S \subset U$ with an accumulation point in U such that $f(z) = g(z)$ for $z \in S$, then $f \equiv g$ in U.*

Proof. (i) Let Z denote the zeros of f. Let A denote the set of accumulation points of Z that lie in U. Since Z is a relatively closed subset of U, and f is not constantly zero, we have $A \subset Z \subsetneqq U$. First we show that A is relatively closed in U. Suppose (z_n) is a sequence in A that tends to $z_0 \in U$. If z_0 equals some z_n, then $z_0 \in A$; if $z_n \neq z_0$ for any n, then (z_n) is a sequence in $Z \setminus \{z_0\}$ that tends to z_0, which implies that z_0 is an accumulation point of Z, i.e., $z_0 \in A$. Thus, A is relatively closed in U. Now we prove that A is open. Recall that for any $z_0 \in U$, there is $r > 0$ such that one of the following three cases occur: 1. $D(z_0, r)$ contains no zero; 2. $D(z_0, r)$ contains only one zero, which is z_0; 3. every point in $D(z_0, r)$ is a zero. In the first two cases z_0 is not an accumulation point of the zeros. Thus, if $z_0 \in A$, then the third case happens for some $r > 0$, which then implies that every point in $D(z_0, r)$ is also an accumulation point of the zeros, i.e., $D(z_0, r) \subset A$. Thus A is an open set. So A is relatively open in U. Since U is connected and $A \subsetneqq U$, we must have $A = \emptyset$. The proof is done.

(ii) Let $h = f - g$. Then h is analytic in U, and S is a subset of the zeros of h. Since S has an accumulation point in U, from (i) we see that h is constant 0, so $f = g$ on U. $\square$

Remarks.

1. We will show later that a holomorphic function is also analytic. So the above theorem also works for holomorphic functions.

2. Uniqueness theorem tells us that if f is analytic on a domain U, and S is a subset of U that contains an accumulation point in U, then the values of f on S determine the whole f.

3. The theorem shows that there is only one way to extend the function e^x from $\mathbb{R}$ to $\mathbb{C}$ such that the new function is holomorphic. Let $f(z) = e^z$. Suppose g is another holomorphic function on $\mathbb{C}$ that satisfies $g(x) = e^x$ for $x \in \mathbb{R}$. Then $f = g$ on $\mathbb{R}$. Since $\mathbb{R}$ has an accumulation point in $\mathbb{C}$, the theorem implies that $f = g$ on $\mathbb{C}$.

4. The above argument also works for the trigonometric functions such as $\sin z$ and $\cos z$. If a trigonometric equality holds for real numbers, then it usually also holds for complex numbers. For example, we now have another method to show that $\cos^2 z + \sin^2 z = 1$. Note that $f(z) := \cos^2 z + \sin^2 z$ is an entire function, and equals $g(z) := 1$ for $z \in \mathbb{R}$. The theorem shows that $f = g$ on $\mathbb{C}$.

Homework.

1. Let f be an analytic function in an open set U. Let $V = \{z \in \mathbb{C} : \overline{z} \in U\}$. Define g on V by $g(z) = \overline{f(\overline{z})}$. Show that g is analytic on V. Note: So far it is not sufficient to show that g is holomorphic because we haven't proved that holomorphic implies analytic.

2. Let U be a nonempty connected open set such that for every $z \in U$, $\overline{z} \in U$. (i) Show that $U \cap \mathbb{R}$ contains an open interval. (ii) Let f be analytic on U. Suppose $f(x) \in \mathbb{R}$ for every $x \in U \cap \mathbb{R}$. Prove that $f(\overline{z}) = \overline{f(z)}$ for any $z \in U$.

3. Prove that there does not exist a function f, which is analytic in $\mathbb{C}$, and satisfies $f(\frac{1}{n}) = |\frac{1}{n^3}|$ for any $n \in \mathbb{Z} \setminus \{0\}$.

Chapter 3

Cauchy's Theorem

3.1 Curves

A curve γ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $[a, b]$ is a real interval. We call $\gamma(a)$ the beginning point, and $\gamma(b)$ the end point of the curve. The set $\{\gamma(t) : a \le t \le b\}$ is called the image of γ. Sometimes a curve may refer to its image set. The reverse of γ is a curve $\gamma^- : [a, b] \to \mathbb{C}$ defined by $\gamma^-(t) = \gamma(a + b - t)$, which has the same image as γ, but the beginning point and end point are swapped. A curve is called closed if its beginning point is the same as its end point.

We say that a curve γ is C^1 if its real part $\operatorname{Re}\gamma$ and imaginary part $\operatorname{Im}\gamma$ both have continuous derivatives (at the endpoints of the interval we consider one-sided derivatives). The derivative of γ is defined by

$$\gamma'(t) = (\operatorname{Re}\gamma)'(t) + i(\operatorname{Im}\gamma)'(t).$$

A curve γ defined on $[a, b]$ is called piecewise C^1 if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that the restriction of γ to every subinterval $[x_{k-1}, x_k]$, $1 \le k \le n$, is C^1. At the partition points, γ has one-sided derivatives in both directions, which may not agree.

From now on, a curve is always assumed to be piecewise C^1 unless otherwise stated. Let γ be a (piecewise C^1) curve. The length of γ is defined by

$$L(\gamma) = \int_a^b |\gamma'(t)|dt = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |\gamma'(t)|dt.$$

The Riemann integral is well defined although $|\gamma'|$ may not be defined at the partition points. We have $L(\gamma^-) = L(\gamma)$ since

$$L(\gamma^-) = \int_a^b |-\gamma'(a + b - t)|dt = \int_a^b |\gamma'(a + b - t)|dt = \int_a^b |\gamma'(t)|dt = L(\gamma).$$

Examples.

1. Let $z_0, w_0 \in \mathbb{C}$. Let $\gamma(t) = (1-t)z_0 + tw_0$, $0 \le t \le 1$. The beginning point is $\gamma(0) = z_0$, and the end point is $\gamma(1) = w_0$. We have $\gamma'(t) = w_0 - z_0$, $0 \le t \le 1$. The image of γ is a line segment connecting z_0 and w_0. The length of γ is $\int_0^1 |\gamma'(t)|dt = |w_0 - z_0|$. We use $[z_0, w_0]$ to denote this curve. Its reverse is $[w_0, z_0]$.

2. For $z_0 \in \mathbb{C}$ and $r > 0$, define $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$. It is a closed curve since $\gamma(0) = \gamma(2\pi) = z_0 + r$. We have $\gamma'(t) = ire^{it}$, $0 \le t \le 2\pi$. The image of γ is a circle: $\{|z - z_0| = r\}$. The length of γ is $\int_0^{2\pi} |\gamma'(t)|dt = 2\pi r$. Later when we view $\{|z - z_0| = r\}$ as a curve, it always means the above γ.

Suppose two (piecewise C^1) curves $\gamma : [a, b] \to \mathbb{C}$ and $\eta : [c, d] \to \mathbb{C}$ satisfy that the endpoint of γ agrees with the initial point of η, i.e., $\gamma(b) = \eta(c)$. Define $\gamma \oplus \eta$ on $[a + c, b + d]$ such that $\gamma \oplus \eta(t) = \gamma(t - c)$, $a + c \le t \le b + c$; and $\gamma \oplus \eta(t) = \eta(t - b)$, $b + c \le t \le b + d$. Then $\gamma \oplus \eta$ is also a (piecewise C^1) curve. The beginning point of $\gamma \oplus \eta$ is the beginning point of γ; the end point of $\gamma \oplus \eta$ is the end point of η; the image of $\gamma \oplus \eta$ is the union of the two images; and the length of $\gamma \oplus \eta$ is the sum of the two lengths. The composition satisfies the associative law, so we may define $\gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_n$ if the endpoint of γ_k agrees with the initial point of γ_{k+1}, $1 \le k \le n - 1$.

Example.

1. If two curves γ and η have the same beginning point and the same end point, then $\gamma \oplus \eta^-$ is a closed curve.

2. Let $z_0, z_1, \dots, z_n \in \mathbb{C}$. Then $[z_0, z_1] \oplus [z_1, z_2] \cdots \oplus [z_{n-1}, z_n]$ is a curve called a polygonal curve. It is closed if $z_n = z_0$.

Let γ be a curve defined on $[a, b]$. Let $\psi : [c, d] \to [a, b]$ be continuously differentiable such that $\psi' > 0$, $\psi(c) = a$, and $\psi(d) = b$. Then $\gamma \circ \psi$ is also a curve, and

$$(\gamma \circ \psi)'(t) = \gamma'(\psi(t))\psi'(t).$$

We say that $\gamma \circ \psi$ is a reparametrization of γ. Note that $\gamma \circ \psi$ and γ have the same beginning point, the same end point, the same image, and the same length. Indeed,

$$\int_c^d |(\gamma \circ \psi)'(t)|dt = \int_c^d |\gamma'(\psi(t))||\psi'(t)|dt = \int_a^b |\gamma'(t)|dt.$$

For example, there are different definitions of $\gamma \oplus \eta$ in the literature, but they are just reparametrization of each other.

Finally, suppose γ lies in an open set U, and f is holomorphic on U, then $f \circ \gamma : [a, b] \to \mathbb{C}$ is also a curve, and

$$(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t).$$

Recall that a domain D is a nonempty connected open set, and every two points $z, w \in D$ can be connected by a continuous curve in D. This result can be improved. It is not hard to prove that there is a C^1 curve in D, whose beginning point is z, and whose end point is w.

Theorem 3.1.1. *Suppose f is holomorphic on a domain U such that $f' = 0$ on U. Then f is a constant.*

Proof. It suffices to show that for any $z_0, w_0 \in U$, $f(z_0) = f(w_0)$. Since U is connected, there is a C^1 curve $\gamma : [a, b] \to U$, which starts from z_0 and ends at w_0. Then $(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t) = 0$ for $a \le t \le b$, which implies that $f(z_0) = f(\gamma(a)) = f(\gamma(b)) = f(w_0)$. □

If f is a function on an open set U and g is a holomorphic function on U such that $g' = f$, then we say that g is a primitive of f on U. The above theorem implies that, if U is connected, a primitive of f is unique up to a constant. That is, if g_1 and g_2 are both primitives of f, then $g_1 - g_2$ is a constant because $(g_1 - g_2)' = g_1' - g_2' = 0$.

3.2 Integrals over curves

Let $F : [a, b] \to \mathbb{C}$ be continuous. Write $F(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real valued functions. The integral of F on $[a, b]$ is defined by

$$\int_a^b F(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt,$$

where the integrals on the righthand side are the usual Riemann integral for real functions. One may also use the limit of Riemann sums to define the integral of F.

It is easy to check the linear property of the above integral. Suppose F and G are complex valued continuous function on $[a, b]$, and $C \in \mathbb{C}$. Then

$$\int_a^b (F + G)(t)dt = \int_a^b F(t)dt + \int_a^b G(t)dt;$$

$$\int_a^b CF(t)dt = C\int_a^b F(t)dt.$$

Let $L = \int_a^b F(t)dt$. Writing L in its polar form, we can find $C \in \mathbb{C}$ with $|C| = 1$ such that $CL = |L|$, which implies that $\int_a^b CF(t)dt = |L|$. Thus,

$$|L| = \int_a^b CF(t)dt = \mathrm{Re}\int_a^b CF(t)dt = \int_a^b \mathrm{Re}(CF(t))dt \le \int_a^b |CF(t)|dt = \int_a^b |F(t)|dt.$$

This implies that

$$\Big|\int_a^b F(t)dt\Big| \le \int_a^b |F(t)|dt.$$

Let f be a continuous function on an open set U. Let $\gamma : [a, b] \to U$ be a (piecewise C^1) curve. We define the integral of f over γ to be

$$\int_\gamma f = \int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(\gamma(t))\gamma'(t)dt.$$

There $a = x_0 < x_1 < \cdots < x_n = b$ are partition points such that γ is C^1 on $[x_{k-1}, x_k]$.

Example.

1. Compute $\int_{\{|z|=r\}} \overline{z}$.
   Recall that $\{|z| = r\}$ is the curve $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$, and $\gamma'(t) = ire^{it}$. Thus,
$$\int_{\{|z|=r\}} \overline{z} = \int_0^{2\pi} \overline{\gamma(t)}\gamma'(t)dt = \int_0^{2\pi} re^{-it}ire^{it}dt = 2\pi i r^2.$$

2. Compute $\int_{[1,1+i]} \frac{1}{z}\, dz$.
   Recall that $[1, 1+i]$ is the curve $\gamma(t) = 1 + it$, $0 \le t \le 1$, and $\gamma'(t) = i$. So we have
$$\int_{[1,1+i]} \frac{1}{z}\, dz = \int_0^1 \frac{i}{1+it}\, dt = \int_0^1 \frac{t+i}{1+t^2}\, dt$$
$$= \int_0^1 \frac{t}{1+t^2}\, dt + i\int_0^1 \frac{1}{1+t^2}\, dt = \frac{1}{2}\log(1+t^2)\Big|_0^1 + i\arctan(t)\Big|_0^1 = \frac{1}{2}\log(2) + i\frac{\pi}{4}.$$
   At this moment we can not write $\int_{[1,1+i]} \frac{1}{z}\, dz = \log(z)|_1^{1+i}$ because log is multi-valued.

We have the following two simple facts.

1. The integral over a curve does not change if the curve is reparameterized. Suppose γ is defined on $[a, b]$ and $\psi : [c, d] \to [a, b]$ is a C^1 function with $\psi(c) = a$ and $\psi(d) = b$. Then
$$\int_{\gamma\circ\psi} f = \int_c^d f(\gamma(\psi(t)))(\gamma\circ\psi)'(t)dt$$
$$= \int_c^d f(\gamma(\psi(t)))\gamma'(\psi(t))\psi'(t)dt = \int_a^b f(\gamma(s))\gamma'(s)ds = \int_\gamma f.$$

2. The integral over the reversal of a curve is the opposite of the original integral. In fact,
$$\int_{\gamma^-} f = \int_a^b f(\gamma^-(t))(\gamma^-)'(t)dt = \int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))dt$$
$$= \int_b^a f(\gamma(s))\gamma'(s)ds = -\int_a^b f(\gamma(s))\gamma'(s)ds = -\int_\gamma f.$$

3. $\int_{\gamma_1\oplus\gamma_2\oplus\cdots\oplus\gamma_n} f = \sum_{k=1}^n \int_{\gamma_k} f$.

Theorem 3.2.1. *Let f be continuous on an open set U, and suppose that f has a primitive g in U. Let γ be a curve in U, which starts from α and ends at β. Then*
$$\int_\gamma f = g(\beta) - g(\alpha).$$

Proof. Suppose γ is defined on $[a, b]$. Then

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b g'(\gamma(t))\gamma'(t)dt = \int_a^b (g \circ \gamma)'(t)dt$$
$$= (g \circ \gamma)(b) - (g \circ \gamma)(a) = g(\beta) - g(\alpha).$$

□

This theorem provides a simpler way to calculate the integral if a premitive is known. In particular, if a premitive exists, then $\int_\gamma f = 0$ whenever γ is closed. If the premitive does not exist, or is unknown, we still have to use the definition to compute the integral.

Examples.

1. Let $U = \mathbb{C} \setminus \{0\}$, $\gamma(t) = \{|z| = 1\} \subset U$, and $f(z) = z^n$, $n \in \mathbb{Z}$. If $n \neq -1$, then f has a primitive in U, which is $\frac{z^{n+1}}{n+1}$. Since γ is a closed curve in U, we get $\int_\gamma f = 0$. If $n = -1$, we calculate
$$\int_\gamma f = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\, dt = \int_0^{2\pi} idt = 2\pi i \neq 0.$$
So $\frac{1}{z}$ has no primitive in $\mathbb{C} \setminus \{0\}$.

Theorem 3.2.2. *Let f be continuous on a domain U. Then the following are equivalent.*

(i) f has a primitive in U.

(ii) If γ_1 and γ_2 are curves in U that have the same beginning point and the same end point, then $\int_{\gamma_1} f = \int_{\gamma_2} f$.

(iii) If γ is a closed curve in U, then $\int_\gamma f = 0$.

Proof. That (i) implies (iii) follows from the above theorem. Now we show that (iii) implies (ii). Note that $\gamma_1 \oplus \gamma_2^-$ is a closed curve in U. So

$$0 = \int_{\gamma_1 \oplus \gamma_2^-} f = \int_{\gamma_1} f + \int_{\gamma_2^-} f = \int_{\gamma_1} f - \int_{\gamma_2} f.$$

Finally, we show that (ii) implies (i). Fix $z_0 \in U$. Since U is connected, for every $z \in U$ we may find a curve γ_z in U from z_0 to z. Define g on U by

$$g(z) = \int_{\gamma_z} f,$$

From (ii) we know that the value of $g(z)$ does not depend on the choice of γ_z. Suppose $[z, w] \subset U$. Let γ_z be a curve in U from z_0 to z. Then $\gamma_z \oplus [z, w]$ is a curve in U from z_0 to w. So we get

$$g(w) - g(z) = \int_{\gamma_z \oplus [z,w]} f - \int_{\gamma_z} f = \int_{[z,w]} f.$$

The proof is finished by the following lemma. □

Lemma 3.2.1. *Let f and g be defined on an open set U. Suppose f is continuous, and $\int_{[z,w]} f = g(w) - g(z)$ whenever $[z, w] \subset U$. Then g is a primitive of f in U.*

Proof. If $[w, z] \subset U$, we have

$$g(w) - g(z) = \int_{[z,w]} f = \int_0^1 f(z + t(w - z))(w - z)dt,$$

which implies that, if $w \neq z$, then

$$\frac{g(w) - g(z)}{w - z} - f(z) = \int_0^1 [f(z + t(w - z)) - f(z)]dt.$$

Fix $z \in U$. Since U is open, there is $R > 0$ such that $D(z, R) \subset U$. Since f is continuous at z, for any $\varepsilon > 0$, there is $\delta \in (0, R)$ such that if $|w - z| < \delta$, then $|f(w) - f(z)| < \varepsilon$. Now if $|w - z| < \delta$ and $t \in [0, 1]$, then $|z + t(w - z) - z| = t|w - z| \leq |w - z| < \delta$, which implies that $|f(z + t(w - z)) - f(z)| < \varepsilon$, and so

$$\Big|\int_0^1 (f(z + t(w - z)) - f(z))dt\Big| \leq \int_0^1 |f(z + t(w - z)) - f(z)|dt \leq \varepsilon.$$

This implies that $\lim_{w\to z} \frac{g(w)-g(z)}{w-z} = f(z)$. $\square$

Recall the sup norm of f on γ is $\|f\|_\gamma = \sup_{z\in\gamma} |f(z)|$.

Lemma 3.2.2. *If f is continuous on a curve γ, then $|\int_\gamma f| \leq \|f\|_\gamma L(\gamma)$.*

Proof. We have

$$\Big|\int_\gamma f\Big| = \Big|\int_a^b f(\gamma(t))\gamma'(t)dt\Big| \leq \int_a^b |f(\gamma(t))\gamma'(t)|dt \leq \int_a^b \|f\|_\gamma|\gamma'(t)|dt = \|f\|_\gamma L(\gamma).$$

$\square$

Remark. For the f in the lemma, we do not have $|\int_\gamma f| \leq \int_\gamma |f|$. In fact, $\int_\gamma |f|$ may not even be a real number. Also do not confuse this lemma with the inequality $|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$, where f is a continuous complex function defined on $[a, b]$.

Theorem 3.2.3. *(i) Let (f_n) be a sequence of continuous functions on U converging uniformly to f. Let γ be a curve in U. Then*

$$\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f.$$

(ii) If $\sum f_n$ is a series of continuous functions converging uniformly on U, then

$$\int_\gamma \sum f_n = \sum \int_\gamma f_n.$$

Proof. (i) First, f is continuous because it is the uniform limit of continuous functions. So $\int_\gamma f$ makes sense. The first statement is immediate from the inequality

$$\Big|\int_\gamma f_n - \int_\gamma f\Big| = \Big|\int_\gamma (f_n - f)\Big| \le \|f_n - f\|_\gamma L(\gamma) \le \|f_n - f\|_U L(\gamma).$$

Note that $L(\gamma)$ is a finite real number. From $\|f_n - f\| \to 0$, we get $\int_\gamma f_n \to \int_\gamma f$. (ii) This clearly follows from (i) because now the partial sum sequence converges uniformly. $\square$

Now we get another proof of the differentiability of a power series. Suppose $\sum_{n=0}^\infty a_n z^n$ has radius $R > 0$. Define f and g on $D(0,R)$ such that

$$f(z) = \sum_{n=0}^\infty a_n z^n, \quad g(z) = \sum_{n=1}^\infty n a_n z^{n-1}.$$

Let $f_n(z) = a_n z^n$ and $g_n = f_n'$ for $n \ge 0$. Note that $g_0 \equiv 0$ and $g_n(z) = n a_n z^{n-1}$ for $n \ge 1$.

Fix $r \in (0,R)$. Let $z_0, w_0 \in D(0,r)$, and γ be a curve in $D(0,r)$ from z_0 to w_0. Then

$$\int_\gamma g_n = f_n(w_0) - f_n(z_0), \quad n \ge 0.$$

Since $\sum f_n$ and $\sum g_n$ converge to f and g, respectively, uniformly on $D(0,r)$, applying the above theorem, we get $\int_\gamma g = f(w_0) - f(z_0)$. Since this holds for any $z_0, w_0 \in D(0,r)$, we conclude that $f' = g$ on $D(0,r)$. Since this holds for any $r \in (0,R)$, we see that $f' = g$ on $D(0,R)$.

Homework. III, §2: 4 (b,c), 6;
Additional problems:

1. Let $a, b \in \mathbb{C}$ and $c \in [a,b]$. Let f be continuous on $[a,b]$. Use the **definition** to show that

   $$\int_{[a,b]} f = \int_{[a,c]} f + \int_{[c,b]} f.$$

   Note: You should stick to the definition, which gives, e.g., $\int_{[a,b]} f = \int_0^1 f(a + t(b-a)) \cdot (b-a)dt$.

3.3 Goursat's Theorem and Local Primitives

Let $S \subset \mathbb{C}$ be nonempty. We define the diameter of S to be

$$\operatorname{diam}(S) := \sup\{|z - w| : z, w \in S\} \ge 0.$$

Note that $\operatorname{diam}(S) < \infty$ if S is bounded.

In the theorem below, if Δ is a triangle with vertices A, B, C such that $ABCA$ surrounds Δ in the counterclockwise direction, then we write

$$\int_{\partial\Delta} = \int_{[A,B]} + \int_{[B,C]} + \int_{[C,A]}.$$

Theorem 3.3.1. [Goursat's Theorem] *Let f be holomorphic on a closed triangle Δ. This means that f is holomorphic on an open set U that contains Δ. Then $\int_{\partial\Delta} f = 0$.*

Proof. Decompose Δ into four triangles of similar shape: Δ_j, $1 \le j \le 4$, using the middle points of its sides. Then we have

$$\int_{\partial\Delta} f = \sum_{j=1}^{4} \int_{\partial\Delta_j} f.$$

Let $C = |\int_{\partial\Delta} f| \ge 0$. From triangle inequality, there is $j_0 \in \{1, 2, 3, 4\}$ such that

$$\Big|\int_{\partial\Delta_{j_0}} f\Big| \ge \frac{1}{4}\Big|\int_{\partial\Delta} f\Big| = \frac{C}{4}.$$

Let $\Delta^{(1)}$ denote this triangle. Similarly, we may decompose $\Delta^{(1)}$ into four triangles of the similar shape: $\Delta_j^{(1)}$, $1 \le j \le 4$, using the middle points of the sides of $\Delta^{(1)}$. One of them must satisfy

$$\Big|\int_{\partial\Delta_j^{(1)}} f\Big| \ge \frac{1}{4}\Big|\int_{\partial\Delta^{(1)}} f\Big| \ge \frac{|C|}{4^2}.$$

Let this triangle be denoted by $\Delta^{(2)}$. Repeating this sequence, we obtain a sequence of triangles $(\Delta^{(n)})_{n=1}^{\infty}$ such that

1. $\Delta \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \cdots \supset \Delta^{(n)} \supset \Delta^{(n+1)} \supset \cdots$.
2. $L(\partial\Delta^{(n)}) = \frac{1}{2^n} L(\partial\Delta)$, $\operatorname{diam}(\Delta^{(n)}) = \frac{1}{2^n}\operatorname{diam}(\Delta)$, $n \in \mathbb{N}$.
3. $|\int_{\partial\Delta^{(n)}} f| \ge \frac{|C|}{4^n}$, $n \in \mathbb{N}$.

Since all $\Delta^{(n)}$ are nonempty compact sets, we conclude that $\bigcap_{n=1}^{\infty} \Delta^{(n)}$ is nonempty.

Let $z_0 \in \bigcap_{n=1}^{\infty} \Delta^{(n)}$. Then $z_0 \in \Delta$. So f is differentiable at z_0. Define

$$h(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0), & z \ne z_0; \\ 0, & z = z_0. \end{cases}$$

Then h is continuous on Δ and

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + h(z)(z - z_0).$$

Let $P(z) = f(z_0) + f'(z_0)(z - z_0)$. Then P is a polynomial, which has a primitive in $\mathbb{C}$. So we have $\int_{\partial\Delta^{(n)}} P = 0$, $n \in \mathbb{N}$. Thus,

$$\int_{\partial\Delta^{(n)}} f = \int_{\partial\Delta^{(n)}} h(z)(z - z_0)dz, \quad n \in \mathbb{N}.$$

Since $z_0 \in \Delta^{(n)}$, we have $|z - z_0| \le \operatorname{diam}(\Delta^{(n)})$ for each $z \in \partial\Delta^{(n)}$. Thus,

$$\Big| \int_{\partial\Delta^{(n)}} h(z)(z - z_0)dz \Big| \le \|h\|_{\partial\Delta^{(n)}} \operatorname{diam}(\Delta^{(n)}) L(\partial\Delta^{(n)}) \le \|h\|_{\Delta^{(n)}} \frac{\operatorname{diam}(\Delta)}{2^n} \frac{L(\partial\Delta)}{2^n}.$$

Recall that $\|h\|_S = \sup_{z\in S} |h(z)|$. Since $|\int_{\partial\Delta^{(n)}} f| \ge \frac{|C|}{4^n}$, we should have

$$\|h\|_{\Delta^{(n)}} \operatorname{diam}(\Delta) L(\partial\Delta) \ge |C|, \quad n \in \mathbb{N}.$$

Since $\lim_{z\to z_0} h(z) = h(z_0) = 0$, and $\Delta^{(n)} \subset \overline{D}(z_0, \operatorname{diam}(\Delta^{(n)}))$, where $\operatorname{diam}(\Delta^{(n)}) \to 0$, we have $\|h\|_{\Delta^{(n)}} \to 0$, which forces $|C| = 0$, i.e., $C = 0$. □

Definition 3.3.1. *Let U be a nonempty open set. We call U a convex domain if for any $z, w \in U$, $[z, w] \subset U$. We call U a star domain if there is $z_0 \in U$ such that for every $z \in U$, the line segment $[z_0, z]$ lies in U. We call z_0 a center of U.*

Note that

1. Every convex domain is a star domain, where every point can act as a center. So a star domain may have more than one centers.

2. If U is a star domain with a center z_0, and if $z_1, z_2 \in U$ satisfy that $[z_1, z_2] \subset U$, and z_0, z_1, z_2 do not lie on the same line, then the triangle Δ with vertices z_0, z_1, z_2 are contained in U. This is because $\Delta = \bigcup_{z\in[z_1,z_2]} [z_0, z]$.

Theorem 3.3.2. *If f is holomorphic on a star domain U, then f has a primitive in U.*

Proof. Let z_0 be a center of U. First, we show that for any $z_1, z_2 \in U$ with $[z_1, z_2] \subset U$, we have

$$\int_{[z_1,z_2]} f + \int_{[z_2,z_0]} f + \int_{[z_0,z_1]} f = 0. \tag{3.1}$$

Since U is a star domain, the integrals all make sense. If z_0, z_1, z_2 form a triangle Δ, which must be contained in U from the above remark. The equality follows from Goursat theorem because the left hand side is equal to either $\int_{\partial\Delta} f$ or $-\int_{\partial\Delta} f$. If they lie on the same line, then one of them, say z_1, lie on the line segment connecting the other two points. From a homework problem, we get

$$\int_{[z_0,z_2]} f = \int_{[z_0,z_1]} f + \int_{[z_1,z_2]} f,$$

which again implies (3.1). The other cases are similar.

Now define g on U such that

$$g(z) = \int_{[z_0,z]} f.$$

Then g is well defined because $[z_0, z] \subset U$. From (3.1) we find that for any $z, w \in U$ with $[z, w] \subset U$,

$$g(z) - g(w) = \int_{[z_0,z]} f - \int_{[z_0,w]} f = \int_{[w,z]} f.$$

From Lemma 3.2.1 we proved before, we see that g is a primitive of f in U. $\square$

Corollary 3.3.1. *If f is holomorphic on a star domain U, then $\int_\gamma f = 0$ for any closed curve γ in U.*

We will often apply the above theorem to the open discs. If f is holomorphic on an open set U, although we may not have a primitive of f in U, if we restrict f to any open disc D contained in U, then the above theorem implies that there is g holomorphic on D such that $g' = f$ on D. We call such g a local primitive of f.

3.4 Cauchy's Theorem for Jordan Curves

A continuous curve $\gamma : [a, b] \to \mathbb{C}$ is called simple if for any $a \le t_1 < t_2 \le b$, $\gamma(t_1) \ne \gamma(t_2)$. It is called simple closed if the above condition is satisfied except that $\gamma(a) = \gamma(b)$. A simple closed curve is also called a Jordan curve.

Theorem 3.4.1. [Jordan Curve Theorem] *Let γ be a Jordan curve. Then $\mathbb{C} \setminus \gamma$ is a disjoint union of two domains. One of these domains is bounded (the interior) and the other is unbounded (the exterior), and γ is the boundary of each domain.*

The theorem is named after Camille Jordan, who found its first proof. The proof is quite complicated. The interior of a Jordan curve is called the Jordan domain bounded by γ, and is denoted by $\mathrm{Int}(\gamma)$.

We say that a Jordan curve has positive/negative orientation if it is oriented counterclockwise/clockwise. If one travels along the Jordan curve with positive/negative orientation, the interior domain of the curve always lies on his left/right. For example, the circle $\{|z - z_0| = r\}$ parameterized by $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$, has positive orientation.

From now on, a simple curve or Jordan curve is assumed to be piecewise C^1 so that the integrals can be well defined.

Theorem 3.4.2. [Cauchy's Theorem for Jordan Curves] *Let J be a Jordan curve. Suppose f is holomorphic on $\mathrm{Int}(J) \cup J$. Then $\int_J f = 0$.*

Proof. We may assume that J is positively oriented. For otherwise, J^- is a positively oriented Jordan curve, and $\int_J = -\int_{J^-}$. Let U denote the open set on which f is holomorphic. By assumption, $\mathrm{Int}(J) \cup J \subset U$. Note that $\mathrm{Int}(J) \cup J$ is compact because it is both closed and bounded. From a homework problem, there is $r > 0$ such that $D(z_0, r) \subset U$ for any $z_0 \in \text{Int}(J) \cup J$. We may use horizontal lines and vertical lines to divide $\text{Int}(J)$ into finitely many domains, each of which is bounded by a Jordan curve with diameter less than r, say $J_1, \dots, J_n$. Suppose each J_k is also positively oriented. Then $\int_J f = \sum_{k=1}^n \int_{J_k} f$. We now prove that $\int_{J_k} f = 0$ for each k. Fix $z_k \in J_k$. Since $\text{diam}(J_k) < r$, we have $J_k \subset D(z_k, r) \subset U$. Since f is holomorphic on $D(z_k, r)$, which is a star domain, f has a primitive in $D(z_k, r)$. Since J_k is a closed curve in $D(z_k, r)$, we have $\int_{J_k} f = 0$. This finishes the proof. $\square$

Note that Goursat's Theorem is a special case of Cauchy's Theorem. But the proof of Cauchy's Theorem relies on a corollary of Goursat Theorem.

Definition 3.4.1. *A domain U is called a simply connected domain if for any Jordan curve $J \subset U$, we have* $\text{Int}(J) \subset U$.

Intuitively, a simply connected domain is a domain with no holes. Every Jordan domain or star domain (and also convex domain) is simply connected. An annulus $A = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ is not simply connected.

Theorem 3.4.3. *If f is holomorphic on a simply connected domain U, then f has a primitive in U.*

Proof. Fix $z_0 \in U$. For every $z \in U$, we may find a simple polygonal curve γ_z in U that starts from z_0 and ends at z. We define $g(z) = \int_{\gamma_z} f$. we now show that the value of $g(z)$ does not depend on the choice of γ_z. This means that, if γ_1 and γ_2 are two simple polygonal curves in U that both start from z_0 and end at z, we need to prove that $\int_{\gamma_1} f = \int_{\gamma_2} f$. First, we consider a special case when γ_1 and γ_2 only meet at z_0 and z. Then $J := \gamma_1 \oplus \gamma_2^-$ is a Jordan curve in U. Since U is simply connected, $\text{Int}(J) \subset U$. So f is holomorphic on $\text{Int}(J) \cup J$. From the previous theorem,

$$0 = \int_J f = \int_{\gamma_1} f - \int_{\gamma_2} f.$$

Now consider the general cases. We may always find $z_0, z_1, \dots, z_n = z$ on $\gamma_1 \cap \gamma_2$, which appear on both curves in the order from z_0 to z, such that for each $1 \le k \le n$, the subcurves $\gamma_1[z_{k-1}, z_k]$ and $\gamma_2[z_{k-1}, z_k]$ either overlap, or meet only at z_{k-1} and z_k. In both cases, we get

$$\int_{\gamma_1[z_{k-1}, z_k]} f = \int_{\gamma_1[z_{k-1}, z_k]} f.$$

In fact, if two subcurves overlap, then the equality holds trivially; if they meet at only two points, then the equality follows from the above argument. Thus, we have

$$\int_{\gamma_1} f = \sum_{k=1}^n \int_{\gamma_1[z_{k-1}, z_k]} f = \sum_{k=1}^n \int_{\gamma_2[z_{k-1}, z_k]} f = \int_{\gamma_2} f.$$

So g is well defined. From Lemma 3.2.1, it now suffices to show that, whenever $[z, w] \subset U$, we have $g(w) - g(z) = \int_{[z,w]} f$. In fact, there is a simple curve γ in U that starts from z_0 and ends at z, which intersects $[z,w]$ only at z. Then the combination $\beta = \gamma \oplus [z,w]$ is a simple curve in U, which starts from z_0 and ends at w. Thus,

$$g(w) - g(z) = \int_\beta f - \int_\gamma f = \int_{[z,w]} f.$$

$\square$

Homework.

1. Prove that Cauchy's theorem for Jordan curves follows from Green's theorem if we assume that f' is continuous.
   Remark: In the definition of holomorphic functions, there is no assumption that f' is continuous. The statement of Cauchy's Theorem is weaker with this assumption.

2. Let γ be a positively oriented Jordan curve. Use Green's Theorem to compute $\int_\gamma \overline{z} dz$.

3.5 Cauchy's Formula

Suppose J_k, $0 \le k \le n$, are disjoint Jordan curves such that $\mathrm{Int}(J_k) \cup J_k$, $1 \le k \le n$, are mutually disjoint, and all contained in $\mathrm{Int}(J_0)$. Then these curves bound a domain U, which is obtained by removing $\mathrm{Int}(J_k) \cup J_k$, $1 \le k \le n$, from $\mathrm{Int}(J_0)$. In other words, U lies inside J_0 and outside J_k, $1 \le k \le n$.

Theorem 3.5.1. *Suppose every J_k has positive orientation. If f is holomorphic on $\overline{U} = U \cup \bigcup_{k=0}^n J_k$, then*

$$\int_{J_0} f = \sum_{k=1}^n \int_{J_k} f.$$

Proof. We may divide U into two domains bounded by Jordan curves C_1 and C_2 with positive orientation. One may check that

$$\int_{C_1} + \int_{C_2} = \int_{J_0} + \sum_{k=1}^n \int_{J_k^-} = \int_{J_0} - \sum_{k=1}^n \int_{J_k}.$$

From Cauchy's theorem for Jordan curves, $\int_{C_j} f = 0$, $j = 1, 2$, which finishes the proof. $\square$

Theorem 3.5.2. [Cauchy's Formula for Jordan Curves] *Let J be a positively oriented Jordan curve. Suppose f is holomorphic on* $\mathrm{Int}(J) \cup J$. *Then*

$$f(w) = \frac{1}{2\pi i} \int_J \frac{f(z)}{z-w} \, dz, \quad w \in \mathrm{Int}(J). \tag{3.2}$$

Proof. Fix $w \in \text{Int}(J)$. Note that $g(z) := \frac{f(z)}{z-w}$ is not holomorphic on $\text{Int}(J) \cup J$, so we can not apply Cauchy's Theorem to conclude that $\int_J g = 0$. However, we may apply the previous theorem. Let $R > 0$ be such that $D(w,R) \subset \text{Int}(J)$. If $r \in (0,R)$, then $\{|z-w| = r\}$ is a Jordan curve that lies inside J, and g is holomorphic on these two Jordan curves and the domain bounded by them. From a previous theorem, we have

$$\int_J g = \int_{|z-w|=r} g = \int_{|z-w|=r} \frac{f(z)-f(w)}{z-w}\,dz + \int_{|z-w|=r} \frac{f(w)}{z-w}\,dz.$$

Note that the equality holds for any $r \in (0,R)$, and $\int_J g$ does not depend on r. So $\int_J g$ is equal to the limit of the righthand side as $r \to 0$. First, using the parametrization $\gamma(t) = w + re^{it}$, $0 \le t \le 2\pi$, we find that

$$\int_{|z-w|=r} \frac{f(w)}{z-w}\,dz = \int_0^{2\pi} \frac{f(w)}{re^{it}} ire^{it} dt = 2\pi i f(w).$$

Second, since $\lim_{z\to w} \frac{f(z)-f(w)}{z-w} = f'(w)$, there exist $\delta, M > 0$ such that $|\frac{f(z)-f(w)}{z-w}| \le M$ if $|z-w| \le \delta$. Thus, if $r \le \delta$, then

$$\Big|\int_{|z-w|=r} \frac{f(z)-f(w)}{z-w}\,dz\Big| \le ML(\{|z-w| = r\}) = M2\pi r,$$

which tends to 0 as $r \to 0$. This shows that $\int_J g = 2\pi i f(w)$, which proves (3.2). $\square$

Theorem 3.5.3. *Let J and f be as in the previous theorem. Then for any $w_0 \in \text{Int}(J)$,*

$$f(w) = \sum_{n=0}^{\infty} (w-w_0)^n \frac{1}{2\pi i} \int_J \frac{f(z)}{(z-w_0)^{n+1}}\,dz, \quad |w-w_0| < \text{dist}(w_0, J). \tag{3.3}$$

In particular, f is analytic in $\text{Int}(J)$, and we have the following Cauchy's Formula:

$$f^{(n)}(w_0) = \frac{n!}{2\pi i} \int_J \frac{f(z)}{(z-w_0)^{n+1}}\,dz, \quad w_0 \in \text{Int}(J), \quad n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

Proof. Fix $w_0 \in \text{Int}(J)$. Let $R = \text{dist}(w_0, J) > 0$. If $|w-w_0| < R \le |z-w_0|$, then

$$\frac{f(z)}{z-w} = \frac{f(z)}{(z-w_0)-(w-w_0)} = \frac{f(z)/(z-w_0)}{1-(w-w_0)/(z-w_0)} = \frac{f(z)}{z-w_0} \sum_{n=0}^{\infty} \frac{(w-w_0)^n}{(z-w_0)^n}.$$

From (3.2) we see that, if $w \in D(w_0, R)$, then $w \in \text{Int}(J)$, and

$$f(w) = \frac{1}{2\pi i} \int_J \sum_{n=0}^{\infty} \frac{f(z)}{(z-w_0)^{n+1}} (w-w_0)^n\,dz. \tag{3.5}$$

For $z \in J$, we have

$$\left|\frac{f(z)}{(z-w_0)^{n+1}}(w-w_0)^n\right| \le \|f\|_J \frac{|w-w_0|^n}{R^{n+1}}.$$

Since J is compact and f is continuous on J, we have $\|f\|_J < \infty$. Since $|w - w_0| < R$, we have

$$\sum_{n=0}^{\infty} \|f\|_J \frac{|w-w_0|^n}{R^{n+1}} < \infty.$$

Thus, $\sum_{n=0}^{\infty} \frac{f(z)}{(z-w_0)^{n+1}}(w-w_0)^n$ converges uniformly on J, which together with (3.5) implies (3.3). Thus, f is analytic in $\mathrm{Int}(J)$. Finally, since $f^{(n)}(w_0) = n!a_n$ if $f(w) = \sum_{n=0}^{\infty} a_n(w-w_0)^n$ near w_0, we get (3.4). □

Remark. If w_0 lies on the exterior of J, then the right hand side of (3.4) is equal to 0 because the function $\frac{f(z)}{(z-w_0)^{n+1}}$ is holomorphic on $\mathrm{Int}(J) \cup J$, and we may apply Cauchy's Theorem.

Corollary 3.5.1. *If f is holomorphic on an open set U, then f is analytic in U. Moreover, f is infinitely many times complex differentiable, and for every $z_0 \in U$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \tag{3.6}$$

holds for any $z \in D(z_0, \widetilde{R})$, where $\widetilde{R} = \infty$ if $U = \mathbb{C}$; or $\widetilde{R} = \mathrm{dist}(z_0, \partial U)$ if $U \subsetneq \mathbb{C}$.

Proof. The statement follows from the previous theorem by choosing $J = \{|z - z_0| = r\}$ where $r \in (0, R)$, and letting $r \to R$. □

Homework. Compute $\int_\gamma \frac{e^{3z}}{(z-2)^3}\, dz$ for (i) $\gamma = \{|z| = 3\}$ and (ii) $\gamma = \{|z| = 1\}$.

Theorem 3.5.4. [Morera's Theorem] *Suppose f is continuous on a domain U, and satisfies that for any closed curve γ in U, $\int_\gamma f = 0$. Then f is analytic.*

Proof. From a theorem we studied, f has a primitive F in U. Such F is holomorphic, which is also analytic. So $f = F'$ is also analytic in U. □

Remarks.

1. So far we have seen that holomorphic is equivalent to analytic. Thus, if f is complex differentiable in an open set, then it is infinitely many times complex differentiable in that set. This phenomena does not exist in Real Analysis. In the rest of this course, we will use the words "analytic" and "holomorphic" interchangeably.

2. The corollary tells us that the radius of the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$, say R, is greater than or equal to $\widetilde{R} = \mathrm{dist}(z_0, \partial U)$ if $U \subsetneq \mathbb{C}$. The equality may not hold since f could be a restriction of a holomorphic function defined on a bigger domain.

3. If $U = \mathbb{C}$, i.e., f is an entire function, then $R = \widetilde{R} = \infty$ for any z_0. So we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad z_0, z \in \mathbb{C}.$$

4. Now we give a case when we can say that the radius $R = \widetilde{R} = \operatorname{dist}(z_0, \partial U)$. Note that if $R > \widetilde{R}$, then $f|_{D(z_0,\widetilde{R})}$ is the restriction of a holomorphic function on $D(z_0, R)$ to $D(z_0, \widetilde{R})$. Since $\overline{D}(z_0, \widetilde{R})$ is a compact subset of $D(z_0, R)$, we then conclude that f is bounded on $D(z_0, \widetilde{R})$. Thus, if there is a point $z_1 \in \partial D(z_0, \widetilde{R})$ such that $|f(z)| \to \infty$ as $z \to z_1$, then we must have $R = \widetilde{R} = \operatorname{dist}(z_0, \partial D)$.

Theorem 3.5.5. [Liouville's Theorem] *Every bounded entire function is constant.*

Proof. Let f be an entire function. Suppose that there is $M \in \mathbb{R}$ such that $|f(z)| \le M$ for any $z \in \mathbb{C}$. Then for any $z \in \mathbb{C}$ and $R > 0$

$$f'(z) = \frac{1}{2\pi i} \int_{|w-z|=R} \frac{f(w)}{(w-z)^2}\, dw.$$

For $w \in \{|w - z| = R\}$, $|\frac{f(w)}{(w-z)^2}| = \frac{|f(w)|}{R^2} \le \frac{M}{R^2}$. Thus,

$$|f'(z)| \le \frac{1}{2\pi} \frac{M}{R^2} L(\{|z - w| = R\}) = \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}.$$

Since this holds for any $R > 0$, we get $f'(z) = 0$ for any $z \in \mathbb{C}$. Thus, f is constant. □

Theorem 3.5.6. [Fundamental Theorem of Algebra] *Every non-constant complex polynomial has a zero in $\mathbb{C}$.*

Proof. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a non-constant polynomial with $a_n \neq 0$. Suppose P has no zero in $\mathbb{C}$. Then $Q(z) = 1/P(z)$ is holomorphic on $\mathbb{C}$. We will show that Q is bounded on $\mathbb{C}$. This requires a careful estimation. We find that

$$\frac{P(z)}{a_n z^n} = 1 + \frac{a_{n-1}}{a_n} z^{-1} + \cdots + \frac{a_0}{z^n} \to 1,$$

as $|z| \to \infty$. Thus, as $|z| \to \infty$, $|P(z)| \to \infty$, which implies that $Q(z) \to 0$. So there is $R > 0$ such $|Q(z)| \le 1$ if $|z| > R$. Since Q is continuous on the compact set $\overline{D}(0, R)$, it is bounded on this set. So there is $M_0 > 0$ such that $|Q(z)| \le M_0$ if $|z| \le R$. Let $M = \max\{M_0, 1\}$. Then $|Q(z)| \le M$ for any $z \in \mathbb{C}$. So Q is a bounded entire function. Applying Liouville's theorem, we see that Q is constant, which implies that P is constant, which is a contradiction. □

We may apply FTA to conclude that every polynomial can be factorized into $C\prod_{k=1}^{n}(z - z_k)$. Suppose z_1 is a zero of $P(z)$, then

$$P(z) = P(z) - P(z_1) = \sum_{k=0}^{n} a_k (z^k - z_1^k) = (z - z_1) \sum_{k=1}^{n} a_k (z^{k-1} + \cdots z_1^{k-1}) =: (z - z_1) Q(z).$$

Note that Q is a polynomial of degree $n-1$. If $\deg(Q)=0$, then Q is constant; if $\deg(Q)\ge 1$, we may find a zero of Q and factorize Q. The conclusion follows from an induction.

At the end, we give a quick review of the development of recent important theorems. All magics begin with Goursat's Theorem, which is about the integral of a holomorphic function over the boundary of a triangle. In the proof we divide the triangle into 4 smaller triangles of similar shapes using middle points, and choose one of them to divide into even smaller triangles. Repeating this process, we get a decreasing sequence of triangles, and then we look at the intersection of them. Goursat's Theorem is then used to prove a corollary about the existence of local primitive of a holomorphic function. Later we prove the Cauchy's Theorem for Jordan curves, which is about the integral of a holomorphic function over a Jordan curve. In the proof we divide the Jordan domain into several Jordan domains of small diameters and apply the corollary of Goursat's Theorem. Cauchy's Theorem is then used to prove Cauchy's formula, where we consider the integral of $\frac{f(z)}{z-w}$ over a small circle centered at w, and let the radius tend to 0. Then we use Cauchy's formula to prove that a holomorphic function is analytic, and also derive integral expressions of the derivatives of f. Cauchy's formula is also used to prove Liouville's Theorem, which is then used to prove the Fundamental Theorem of Algebra.

Homework III, §7: 3
Additional problems:

1. What is the radius of the power series $\sum_{n=0}^{\infty}\frac{\tanh^{(n)}(0)}{n!}z^n$? Justify your answer.
   Remark. Since tanh restricted to $\mathbb{R}$ is a real analytic function, one may ask this question with only Real Analysis. However, it is hard to answer without Complex Analysis.

2. Let f be an entire function. Suppose that there exists $r>0$ such that $|f(z)|\ge r$ for every $z\in\mathbb{C}$. Prove that f is constant.

3. Let f be an entire function. Suppose that f has two periods $a,b\in\mathbb{C}$, which are $\mathbb{R}$-linearly independent. This means that $f(z+a)=f(z+b)=f(z)$ for any $z\in\mathbb{C}$, and $ax+by\ne 0$ for any $x,y\in\mathbb{R}$ which are not both zero. Prove that f is constant.

3.6 Differentiability of the Logarithm Function

Let U be an open set with $0\notin U$. Recall that $L(z)$ is called a branch of $\log z$ in U, if it is continuous in U, and satisfies $e^{L(z)}=z$ for any $z\in U$. In this section, we will show that a branch of $\log z$ is not only continuous, but also holomorphic, and its derivative is the function $1/z$.

Theorem 3.6.1. *Suppose f is a primitive of $1/z$ in a domain U. Then there is some $C\in\mathbb{C}$ such that $f+C$ is a branch of $\log z$ in U.*

Proof. Let $g(z)=\frac{e^{f(z)}}{z}$. Then g is holomorphic on U. We compute

$$g'(z)=\frac{e^{f(z)}f'(z)}{z}-\frac{e^{f(z)}}{z^2}=\frac{e^{f(z)}}{z^2}-\frac{e^{f(z)}}{z^2}=0,\quad z\in U.$$

Since U is connected, g is constant C_0, which is not 0. Thus, $e^{f(z)} = C_0 z$ for any $z \in U$. Let $C \in \mathbb{C}$ be such that $e^C = 1/C_0$. Then $e^{f(z)+C} = z$ in U. Since f is holomorphic on U, $f + C$ is continuous in U, and so is a branch of $\log z$ in U. □

Theorem 3.6.2. *Let $U \subset \mathbb{C} \setminus \{0\}$ be a simply connected domain. Then there is a branch of $\log z$ in U, which is a primitive of $1/z$ in U.*

Proof. Since $1/z$ is holomorphic on U, which has a primitive in U as U is simply connected. Let f denote this primitive. From the above theorem, there is a constant C such that $g := f + C$ is a branch of $\log z$. The g is what we need since $g' = f' = 1/z$ in U. □

Lemma 3.6.1. *If f is a continuous function on a domain U such that $f(z) \in \mathbb{Z}$ for every $z \in U$, then f is constant.*

Proof. Let $z_0 \in U$ and $n_0 = f(z_0) \in \mathbb{Z}$. Let $A = f^{-1}(\{n_0\}) \ni z_0$. Since f is continuous on U, A is relatively closed in U. On the other hand, since f only takes integer values, $A = f^{-1}((n_0 - 1/2, n_0 + 1/2))$. Since f is continuous on U, A is relatively open in U. Since U is connected, and A is not empty, we have $A = U$. So f is constant n_0. □

Theorem 3.6.3. *If L is a branch of $\log z$ in an open set $U \subset \mathbb{C} \setminus \{0\}$, then L is a primitive of $1/z$ in U.*

Proof. Let $z_0 \in U$. Let $r > 0$ be such that $D(z_0, r) \subset U$. Since $D(z_0, r)$ is simply connected, from Theorem 3.6.2, there is a branch $M(z)$ of $\log z$ in $D(z_0, r)$, which is a primitive of $1/z$ in that disc. Since $L(z)$ and $M(z)$ are both branches of $\log z$ in $D(z_0, r)$, we find that $f(z) := (L(z) - M(z))/(2\pi i)$ is an integer valued continuous function on the disc. From the above lemma, f is constant. So $L' = M' = 1/z$ in $D(z_0, r)$. Since $z_0 \in U$ is arbitrary, we conclude that L is a primitive of $1/z$ in U. □

Let f be an analytic function in an open set U, which does not take value 0. We say that g is a branch of $\log f$ in U, if g is continuous and satisfies that $e^g = f$. Note that if g is holomorphic, then $e^g g' = f'$, which gives $g' = \frac{f'}{f}$. Using similar proofs, we can prove the following statements in the order:

1. Suppose h is a primitive of $\frac{f'}{f}$ in a domain U. Then there is some $C \in \mathbb{C}$ such that $h + C$ is a branch of $\log f$ in U. For this statement, note that

$$\frac{d}{dz}\frac{e^{h(z)}}{f(z)} = \frac{e^{h(z)}h'(z)}{f(z)} - \frac{e^{h(z)}f'(z)}{f(z)^2} = 0.$$

2. If U is a simply connected domain, then there exists a branch of $\log f$, which is also a primitive of $\frac{f'}{f}$.

3. For general U, if g is a branch of $\log f$, then g is a primitive of $\frac{f'}{f}$.

Homework. Chapter III, §6: 6.
Additional problems.

1. Fix $\alpha \in \mathbb{C}$. The function $(1+z)^\alpha$ in $D(0,1)$ is defined as $e^{\alpha L(z)}$, where L is a branch of $\log(1+z)$ that satisfies $L(0)=0$. Prove that $(1+z)^\alpha$ is holomorphic on $D(0,1)$, and
$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad z \in D(0,1),$$
where $\binom{\alpha}{n}$ was defined earlier.

2. Let g be holomorphic on a simply connected domain U. Show that there is f, which is holomorphic on U without zero, such that $g = \frac{f'}{f}$ in U.

3.7 The Maximum Modulus Principle

Theorem 3.7.1. [Mean Value Theorem] *Let f be holomorphic on a closed disc $\overline{D}(z_0,r)$. Then*
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta;$$
$$f(z_0) = \frac{1}{\pi r^2}\int\int_{|z-z_0|\le r} f(z)dxdy. \tag{3.7}$$

Proof. From Cauchy's Formula,
$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta.$$
This is also true if r is replaced by any $s \in (0,r)$. Thus,
$$2\pi s f(z_0) = \int_0^{2\pi} f(z_0+se^{i\theta})sd\theta, \quad 0 \le s \le r.$$
Integrating s from 0 to r, we get
$$\pi r^2 f(z_0) = \int_0^{2\pi}\int_0^r f(z_0+se^{i\theta})sdsd\theta = \int\int_{|z-z_0|\le r} f(z)dxdy,$$
where in the last step we used $rdrd\theta = dxdy$. □

Theorem 3.7.2. *Let f be holomorphic on $\overline{D}(z_0,r)$. Suppose $|f(z_0)| \ge |f(z)|$ for any $z \in D(z_0,r)$. Then f is constant on $D(z_0,r)$.*

Proof. First, we show that $|f|$ is constant on $D(z_0, r)$. If not, there is $z_1 \in D(z_0, r)$ such that $|f(z_1)| < |f(z_0)|$. Let $\varepsilon = |f(z_0)| - |f(z_1)| > 0$. Since $|f|$ is continuous, we may find $r_1 > 0$ such that $D(z_1, r_1) \subset D(z_0, r)$ and $|f(z)| \le |f(z_1)| + \varepsilon/2 = |f(z_0)| - \varepsilon/2$ for $z \in D(z_1, r_1)$. Let $D_0 = D(z_0, r)$ and $D_1 = D(z_1, r_1)$. From (3.7), we get

$$\pi r^2 |f(z_0)| \le \int\int_{D_0} |f(z)| dxdy = \int\int_{D_1} |f(z)| dxdy + \int\int_{D_0 \setminus D_1} |f(z)| dxdy$$

$$\le \int\int_{D_1} (|f(z_0)| - \varepsilon/2) dxdy + \int\int_{D_0 \setminus D_1} |f(z_0)| dxdy = \int\int_{D_0} |f(z_0)| dxdy - \frac{\varepsilon}{2}\pi r_1^2 < \pi r^2 |f(z_0)|,$$

which is a contradiction. Thus, $|f|$ is constant in $D(z_0, r)$. Using a homework problem, we then conclude that f is constant in $D(z_0, r)$. □

Corollary 3.7.1. *Let f be holomorphic on a domain U. Suppose $|f|$ attains local maximum at some $z_0 \in U$, i.e., there is $r > 0$ such that $|f(z_0)| \ge |f(z)|$ for any $z \in D(z_0, r)$. Then f is constant in U.*

Proof. From the previous two theorems, we see that f is constant in $D(z_0, r)$. This means that z_0 is an accumulation point of the zeros of $f - f(z_0)$. Since U is a connected, $f - f(z_0)$ is constant 0 in U. So f is constant in U. □

Theorem 3.7.3. [**Maximum Modulus Principle**] *Let U be a bounded domain and $\overline{U}$ be its closure. Suppose f is a continuous on $\overline{U}$, and holomorphic on U. Then there is $z_0 \in \partial U$ such that $|f(z_0)| \ge |f(z)|$ for any $z \in \overline{U}$. In short, the maximum of $|f|$ is attained at the boundary.*

Proof. Since U is bounded, $\overline{U}$ is a compact set. Since $|f|$ is continuous on $\overline{U}$, it attains its maximum at some $w_0 \in \overline{U}$. If $w_0 \in \partial U$, then the proof is done by taking $z_0 = w_0$. If $w_0 \in U$, then from the above corollary, f is constant in U. From the continuity, that f is also constant in $\overline{U}$. In this case, any $z_0 \in \partial U$ works. □

Corollary 3.7.2. *Let U be a bounded domain. Suppose that both f and g are continuous on $\overline{U}$, and holomorphic on U. If $f = g$ on ∂U, then $f = g$ in $\overline{U}$.*

Proof. Let $h = f - g$. Then h is continuous on $\overline{U}$, and holomorphic on U. If $f = g$ on ∂U, then $|h| = 0$ on ∂U. The above corollary implies that $h = 0$ in $\overline{U}$, i.e., $f = g$ in $\overline{U}$. □

Remark. The condition that f is a continuous on $\overline{U}$ and holomorphic on U is often satisfied when f is holomorphic on $\overline{U}$, i.e., f is holomorphic on another open set U' with $U' \supset \overline{U}$.

Homework III, §1: 2, 3

3.8 Harmonic Functions

Let $U \subset \mathbb{R}^n$ be an open set. Let $f : U \to \mathbb{C}$. If f is C^2 on U, and satisfies the Laplace equation

$$\Delta f(x) := \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}(x) = 0, \quad x \in U,$$

then we say that f is a harmonic function on U. The symbol Δ is called the Laplace operator.

In this course, we focus on the case $n = 2$, and identify $\mathbb{R}^2$ with $\mathbb{C}$. The Laplace equation becomes

$$\Delta f(z) = \frac{\partial^2 f}{\partial x^2}(z) + \frac{\partial^2 f}{\partial y^2}(z) = 0, \quad z \in U.$$

Note that a complex function is harmonic if and only if both of its real part and imaginary part are harmonic.

Theorem 3.8.1. *Let f be holomorphic on an open set $U \subset \mathbb{C}$. Then f is harmonic on U.*

Proof. Let $f = u+iv$. We have seen that f is infinitely many times complex differentiable, which implies that u and v are infinitely many times real differentiable. From the Cauchy-Riemann equation, we get $u_x = v_y$ and $u_y = -v_x$ in U. Thus,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0, \quad v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0,$$

which implies that both u and v are harmonic, and so is f. $\square$

From now on, we assume that a harmonic function is always real valued.

Lemma 3.8.1. *Let u be a real valued C^2 function defined in an open set U. Then u is harmonic on U if and only if $u_x - iu_y$ is holomorphic on U.*

Proof. Suppose u is harmonic on U. Then $u_x, u_y \in C^1$ and $(u_x)_x = (-u_y)_y$ and $(u_x)y = -(-u_y)_x$. Cauchy-Riemann equation is satisfied by u_x and $-u_y$. So $u_x - iu_y$ is holomorphic. On the other hand, if $u_x - iu_y$ is holomorphic, then the Cauchy-Riemann equation implies that $(u_x)_x = (-u_y)_y$, i.e., $u_{xx} + u_{yy} = 0$. So u is harmonic. $\square$

Definition 3.8.1. *Let u be a harmonic function in a domain D. If a real valued function v satisfies that $u + iv$ is holomorphic on D, then we say that v is a harmonic conjugate of u in D.*

A harmonic conjugate must also be a harmonic function because it is the imaginary part of a holomorphic function. If v and w are both harmonic conjugates of u in U, then $v_x = -u_y = w_x$ and $v_y = u_x = w_y$ in U. Since U is connected, we get $v - w$ is constant. This means that, the harmonic conjugates of a harmonic function, if it exists, are unique up to an additive constant. Also note that if v is a harmonic conjugate of u, then $-u$ (instead of u) is a harmonic conjugate of v. This is because $-i(u + iv) = v - iu$ is holomorphic.

Theorem 3.8.2. *Let u be a harmonic function in a simply connected domain D. Then there is a harmonic conjugate of u in D.*

Proof. Let $f = u_x - iu_y$ in D. From the above lemma, f is holomorphic on D. Since D is simply connected, f has a primitive in D, say F. Write $F = \widetilde{u} + i\widetilde{v}$. Then

$$u_x - iu_y = f = F' = \widetilde{u}_x - i\widetilde{u}_y.$$

Thus, $u_x = \widetilde{u}_x$ and $u_y = \widetilde{u}_y$ in U. Since D is connected, we see that $\widetilde{u} - u$ is a real constant. Let $C = \widetilde{u} - u \in \mathbb{R}$. Then $F - C = u + i\widetilde{v}$ is holomorphic on D. Thus, $\widetilde{v}$ is a harmonic conjugate of u. $\square$

Remarks.

1. The theorem does not hold if we do not assume that U is simply connected. Here is an example. Let $D = \mathbb{C} \setminus \{0\}$. Let $u(z) = \ln|z| = \frac{1}{2}\ln(x^2 + y^2)$. Then $u_x = \frac{x}{x^2+y^2}$ and $u_y = \frac{y}{x^2+y^2}$. So $u_x - iu_y = \frac{1}{x+iy}$ is holomorphic on D. From the above lemma, u is harmonic. If v is a harmonic conjugate of u in D, then $u + iv$ is a primitive of $u_x - iu_y = \frac{1}{z}$ in D. However, we already know that $\frac{1}{z}$ has no primitive in $\mathbb{C} \setminus \{0\}$. Recall that $\int_{|z|=1} \frac{dz}{z} = 2\pi i \neq 0$. Thus, u has no harmonic conjugates in D.

2. A harmonic conjugate always exists locally: if u is a harmonic function in an open set U, then for any disk $D(z_0, r) \subset U$, there is f, which is holomorphic on $D(z_0, r)$ and satisfies that $\operatorname{Re} f = u$. Since such f is infinitely many times complex differentiable, we see that u is infinitely many times real differentiable in $D(z_0, r)$. Since $D(z_0, r) \subset U$ can be chosen arbitrarily, we see that every harmonic function is infinitely many times real differentiable.

Homework.
Find all real-valued C^2 differentiable functions h defined on $(0, \infty)$ such that $u(x, y) = h(x^2+y^2)$ is harmonic on $\mathbb{C} \setminus \{0\}$.

If U is simply connected, we may use the following method to find a harmonic conjugate of u. Here is an example. Let $u(x, y) = x^2 + 2xy - y^2$. Then $u_{xx} + u_{yy} = 2 - 2 = 0$. So u is harmonic on $\mathbb{R}^2$. We now find a harmonic conjugate of u. If v is a harmonic conjugate, then $v_y = u_x = 2x + 2y$. Thus, $v = 2xy + y^2 + h(x)$, where $h(x)$ is a differentiable function in x. From $-u_y = v_x$, we get $2y - 2x = 2y + h'(x)$. So we may choose $h(x) = -x^2$. So one harmonic conjugate of u is $2xy + y^2 - x^2$.

Theorem 3.8.3. [Mean Value Theorem for Harmonic Functions] *Let u be harmonic on $D(z_0, R)$. Then for any $r \in (0, R)$,*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta;$$

$$u(z_0) = \frac{1}{\pi r^2}\int_{|z-z_0|\le r} u(z)dxdy.$$

Proof. Since $D(z_0, R)$ is simply connected, there is f holomorphic on $D(z_0, R)$ such that $u = \operatorname{Re} f$. From the Mean Value Theorem for holomorphic functions, the two formulas hold with f in place of u. Then we can obtain the formulas for u by taking the real parts. □

Corollary 3.8.1. *With the above setup, if u attains its maximum at z_0, then u is constant in $D(z_0, R)$.*

Proof. We have seen a similar proposition, which says that if f is holomorphic on $D(z_0, R)$, and $|f|$ attains its maximum at z_0, then $|f|$ is constant in $D(z_0, R)$. A similar proof can be used here.

Here is another proof. Let f be analytic such that $u = \operatorname{Re} f$. Then e^f is also analytic, and $|e^f| = e^u$. Since u attains its maximum at z_0, $|e^f|$ also attains its maximum at z_0. An earlier proposition shows that $|e^f|$ is constant, which implies that $u = \log |e^f|$ is constant. □

Theorem 3.8.4. **[Maximum Principle for Harmonic Functions]** *Let u be harmonic on a domain D.*

(i) Suppose that u has a local maximum at $z_0 \in D$. Then u is constant.

(ii) If D is bounded, and u is continuous on $\overline{D}$, then there is $z_0 \in \partial D$ such that $u(z_0) = \max\{u(z) : z \in \overline{D}\}$.

(iii) The above statements also hold if "maximum" is replaced by "minimum".

Proof. (i) From the above corollary, there is $r_0 > 0$ such that u is constant in $D(z_0, r_0)$. Let $w \in D$. Since D is connected, we may find a finite sequence of disks $D_k = D(z_k, r_k)$, $0 \le k \le n$, in D, such that $w \in D_n$ and $D_{k-1} \cap D_k \neq \emptyset$, $1 \le k \le n$. Since each D_k is simply connected, there is f_k holomorphic on D_k such that $u = \operatorname{Re} f_k$ in D_k. We already see that u is constant in D_0. So $\operatorname{Re} f_1 = u$ is constant in $D_0 \cap D_1$. From C-R equations, we see that f_1 is constant in $D_0 \cap D_1$. From the Uniqueness Theorem, we see that f_1 is constant in D_1. Thus, $u = \operatorname{Re} f_1$ is constant in D_1. Using induction, we see that u is constant in every D_k. Since $D_{k-1} \cap D_k \neq \emptyset$, u is constant in $\bigcup_{k=0}^n D_k$. Thus, $f(w) = f(z_0)$ as $w \in D_n$ and $z_0 \in D_0$.

(ii) Since D is bounded, $\overline{D}$ is compact. Since u is continuous on $\overline{D}$, it attains its maximum at some $w_0 \in \overline{D}$. If $w_0 \in \partial D$, we may let $z_0 = w_0$. If $w_0 \in D$, then (i) implies that u is constant on D. The continuity then implies that u is constant in $\overline{D}$. We may take z_0 to be any point on ∂D.

(iii) Note that $-u$ is also harmonic, and when $-u$ attains its maximum, u attains its minimum. □

Corollary 3.8.2. *Suppose u and v are both harmonic on a bounded domain D and continuous on $\overline{D}$. Suppose that $u = v$ on ∂D. Then $u = v$ on $\overline{D}$.*

Proof. Let $h = u - v$. Then h is harmonic on D, continuous on $\overline{D}$, and $h \equiv 0$ on ∂U. From the above theorem, h attains its maximum and minimum in $\overline{D}$ at ∂D. So h has to be 0 everywhere in $\overline{D}$, i.e., $u = v$ on $\overline{U}$. □

The above corollary says that, if u is harmonic on a bounded domain D and continuous on $\overline{D}$, then the values of u on D are determined by the values of u on ∂D.

We introduce the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

This mean that, if $f = u + iv$, then

$$f_z := \frac{\partial f}{\partial z} = \frac{1}{2}(u_x + iv_x) - \frac{i}{2}(u_y + iv_y) = \frac{u_x + v_y}{2} + i\frac{v_x - u_y}{2};$$
$$f_{\overline{z}} := \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(u_x + iv_x) + \frac{i}{2}(u_y + iv_y) = \frac{u_x - v_y}{2} + i\frac{v_x + u_y}{2}.$$

So the Cauchy-Riemann equation is equivalent to $f_{\overline{z}} = 0$; and if f is holomorphic, then $f_z = u_x + iv_x = f'$. Moreover, it is clear that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \frac{1}{4}\Delta.$$

Thus, if f is holomorphic, then $\Delta f = 0$, from which we see again that f is harmonic. If u is harmonic, then from $\partial_{\overline{z}}\partial z u = \frac{1}{4}\Delta u = 0$ we see that $\partial_z u$ is holomorphic, which is used in a proof a theorem.

Remark. The smoothness, mean value theorem and the maximum principle also hold for harmonic functions in R^n for $n \geq 3$. But the technique of complex analysis can not be used. For example, the mean value theorem follows from the divergence theorem.

Homework. Chapter VIII, §1: 7 (a,b,c,e).

1. Prove that any positive harmonic function in $\mathbb{R}^2$ is constant. Hint: If f is an entire function with $\operatorname{Re} f > 0$, then consider e^{-f}.
   Remark: This statement does not hold for R^d with $d \geq 3$.

2. Let u be a nonconstant harmonic function on $\mathbb{C}$. Show that for any $c \in \mathbb{R}$, $u^{-1}(c)$ is unbounded. Hint: $\{|z| > R\}$ is connected for any $R > 0$.

3.9 Winding Numbers

Let γ be a closed curve, and $\alpha \in \mathbb{C} \setminus \gamma$. The winding number or index of γ with respect to α is

$$W(\gamma, \alpha) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - \alpha}\, dz.$$

Example. Suppose γ is a Jordan curve. If α lies in the exterior of γ, then applying Cauchy's Theorem to $f(z) = \frac{1}{z-\alpha}$, we get $W(\gamma, \alpha) = 0$. If α lies in the interior of γ, then applying Cauchy's Formula to $f(z) = 1$, we get $W(\gamma, \alpha) = 1$ or -1, where the sign depends on the orientation of γ.

Lemma 3.9.1. *$W(\gamma,\alpha)\in\mathbb{Z}$.*

Proof. Suppose γ is defined on $[a,b]$. Define $F(t)=\int_a^t \frac{\gamma'(s)}{\gamma(s)-\alpha}ds$, $a\le t\le b$. Then F is continuous on $[a,b]$, $F(a)=0$, $F(b)=2\pi i W(\gamma,\alpha)$, and $F'(t)=\frac{\gamma'(t)}{\gamma(t)-\alpha}$ for $t\in[a,b]$ other than the partition points, say $a=x_0<x_1<\cdots<x_n=b$.We now compute

$$\frac{d}{dt}e^{-F(t)}(\gamma(t)-\alpha)=e^{-F(t)}\gamma'(t)-e^{-F(t)}F'(t)(\gamma(t)-\alpha)=0,\quad t\in[a,b]\setminus\{x_0,\dots,x_n\}.$$

Hence there is a constant $C\in\mathbb{C}$ such that $C(\gamma(t)-\alpha)=e^{F(t)}$, $a\le t\le b$. Since γ is closed, we have $e^{F(b)}=e^{F(a)}=e^0=1$, which implies that $F(b)\in 2\pi i\mathbb{Z}$. So $W(\gamma,\alpha)=\frac{1}{2\pi i}F(b)\in\mathbb{Z}$. $\square$

Remark. Let θ_0 be an argument of the C in the above proof. From $\gamma(t)-\alpha=Ce^{F(t)}$ we see that $\operatorname{Im}F(t)+\theta_0$ is an argument of $\gamma(t)-\alpha$ for $a\le t\le b$. Now suppose h is a continuous function on $[a,b]$ such that $h(t)$ is an argument of $\gamma(t)-\alpha$ for $a\le t\le b$, then $(h(t)-\operatorname{Im}F(t)-\theta_0)/(2\pi i)$ is an integer-valued continuous function on $[a,b]$, which must be constant. Thus,

$$W(\gamma,\alpha)=\frac{F(b)-F(a)}{2\pi i}=\frac{i\operatorname{Im}F(b)-i\operatorname{Im}F(a)}{2\pi i}=\frac{h(b)-h(a)}{2\pi}.$$

This means that $2\pi W(\gamma,\alpha)$ equals to the total increment of $\arg(z-\alpha)$ along γ.

Lemma 3.9.2. *The map $\alpha\mapsto W(\gamma,\alpha)$ is continuous on $\mathbb{C}\setminus\gamma$.*

Proof. Fix $\alpha_0\in\mathbb{C}\setminus\gamma$. Let (α_n) be a sequence that converges to α_0. It suffices to show that $\frac{1}{z-\alpha_n}\to\frac{1}{z-\alpha_0}$ uniformly on $z\in\gamma$. Let $r=\operatorname{dist}(\alpha_0,\gamma)>0$. For n big enough, we have $|\alpha_n-\alpha_0|<r/2$, which implies that $\operatorname{dist}(\alpha_n,\gamma)\ge r/2$. For those n, we have

$$\Big|\frac{1}{z-\alpha_n}-\frac{1}{z-\alpha_0}\Big|=\frac{|\alpha_n-\alpha_0|}{|z-\alpha_n||z-\alpha_0|}\le\frac{|\alpha_n-\alpha_0|}{r^2/2},\quad z\in\gamma.$$

Thus, $\|\frac{1}{z-\alpha_n}-\frac{1}{z-\alpha_0}\|_\gamma\le\frac{|\alpha_n-\alpha_0|}{r^2/2}$ when n is big enough, which implies that $2\pi iW(\gamma,\alpha_n)=\int_\gamma\frac{1}{z-\alpha_n}dz\to\int_\gamma\frac{1}{z-\alpha_0}dz=2\pi iW(\gamma,\alpha_0)$. $\square$

Corollary 3.9.1. *$W(\gamma,\cdot)$ is constant on each connected component of $\mathbb{C}\setminus\gamma$.*

Proof. This follows from the above two lemmas and the fact that a continuous integer valued function is constant on a domain. $\square$

Corollary 3.9.2. *$W(\gamma,\alpha)=0$ if α lies on the unbounded component of $\mathbb{C}\setminus\gamma$.*

Proof. Since γ is bounded, there is $M\in(0,\infty)$ such that $|z|\le M$ for all $z\in\gamma$. Suppose $|\alpha|>2M$. Then $|z-\alpha|\ge|\alpha|-|z|\ge\frac{|\alpha|}{2}$ for $z\in\gamma$. Thus, $|\frac{1}{z-\alpha}|\le\frac{2}{|\alpha|}$ if $|\alpha|>2M$. Thus, $|\frac{1}{z-\alpha}|$ tends to 0 as $\alpha\to\infty$ uniformly on $z\in\gamma$. Thus, $W(\gamma,\alpha)\to0$ as $\alpha\to\infty$. Since $W(\gamma,\cdot)$ is constant on the the unbounded component of $\mathbb{C}\setminus\gamma$, the constant has to be 0. $\square$

We define a contour γ to be a "sum" of finitely many closed curves γ_k, $1 \le k \le n$, which may or may not have intersections. The repetitions in γ_k's are allowed. The integral along a contour is defined to be $\int_\gamma = \sum_{k=1}^n \int_{\gamma_k}$. The winding number of a contour γ with respect to $\alpha \in \mathbb{C} \setminus \gamma = \mathbb{C} \setminus \bigcup_{k=1}^n \gamma_k$ is $W(\gamma, \alpha) = \sum_{k=1}^n W(\gamma_k, \alpha)$. The above propositions also hold for contours.

Examples.

1. The winding numbers of a trefoil knot in 5 different domains.

Observe that the winding number increases by 1 if we cross the contour from its right to its left; decreases by 1 if we cross the contour from its left to its right.

Theorem 3.9.1. [The General Cauchy's Theorem] *Let f be holomorphic on a domain U. Let γ be a contour in U such that $W(\gamma, \alpha) = 0$ for every $\alpha \in \mathbb{C} \setminus U$. Then $\int_\gamma f = 0$.*

The interested reader may refer to Chapter IV, § 3 of Lang's book for a proof. The condition that $W(\gamma, \alpha) = 0$ for every $\alpha \in \mathbb{C} \setminus U$ means that every component of $\mathbb{C} \setminus \gamma$ such that the winding number of γ is not zero must be contained in U. For example, if γ is a Jordan curve, then we need that $\mathrm{Int}(\gamma) \subset U$. Suppose now the contour γ is the sum of mutually disjoint Jordan curves γ_k, $0 \le k \le n$, such that $\mathrm{Int}(\gamma_k)$, $1 \le k \le n$, are mutually disjoint, and are all contained in $\mathrm{Int}(\gamma_0)$. Suppose further that γ_0 is positively oriented, and γ_k, $1 \le k \le n$, are negatively oriented. Then the $W(\gamma, \alpha) \ne 0$ iff $\alpha \in D := \mathrm{Int}(\gamma_0) \setminus \cup_{k=1}^n (\mathrm{Int}(\gamma_k) \cup \gamma_k)$. Then the condition of the above theorem becomes $D \subset U$.

Theorem 3.9.2. [The General Cauchy's Formula] *Let f be holomorphic on a domain U. Let γ be a contour in U such that for every $\alpha \in \mathbb{C} \setminus U$, $W(\gamma, \alpha) = 0$. Let $z_0 \in U \setminus \gamma$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = W(\gamma, z_0) f(z_0).$$

Proof. Assuming the general Cauchy's Theorem, the proof of this theorem is not difficult. Let $r > 0$ be such that $\bar{D}(z_0, r) \subset U$. Define a contour η to be $\gamma + (-W(\gamma, z_0))\{|z - z_0| = r\}$. Here if $W(\gamma, z_0) = 0$, then $\eta = \gamma$; if $W(\gamma, z_0) > 0$, this should be understood as $\eta = \gamma + W(\gamma, z_0)\{|z - z_0| = r\}^-$. Let $U' = U \setminus \{z_0\}$. Then for any $\alpha \in \mathbb{C} \setminus U'$, $W(\eta, \alpha) = 0$. Since $\frac{f(z)}{z - z_0}$ is holomorphic on U', from the general Cauchy's Theorem,

$$0 = \frac{1}{2\pi i} \int_\eta \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz - \frac{W(\gamma, z_0)}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{z - z_0}\, dz$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz - W(\gamma, z_0) f(z_0),$$

where the last equality follows from the Cauchy's Formula for Jordan curves. □

Homework. Find the winding numbers for a given closed curve. See the course webpage.

Chapter 4

Calculus of Residues

4.1 Laurent Series

The Laurent series (centered at 0) is of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where a_n, $n \in \mathbb{Z}$, are complex numbers. It converges iff the following two series both converges:

$$\sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=-\infty}^{-1} a_n z^n.$$

The first is a power series. The second can also be transformed into a power series:

$$\sum_{n=-\infty}^{-1} a_n z^n = \sum_{n=1}^{\infty} a_{-n} (1/z)^n.$$

Suppose the radius of $\sum a_n z^n$ is R_+, and the radius of $\sum a_{-n} w^n$ is R_-. Then $\sum_{n=-\infty}^{\infty} a_n z^n$ converges when $|z| < R_+$ and $|1/z| < R_-$, i.e., $1/R_- < |z| < R_+$. Suppose that $1/R_- < R_+$. Let $R = R_+$, $r = 1/R_-$, and let A be the annulus $\{r < |z| < R\}$. Let $f_+(z) = \sum_{n=0}^{\infty} a_n z^n$, $g_-(w) = \sum_{n=1}^{\infty} a_{-n} w^n$, and $f_-(z) = g_-(1/z)$. Then f_+ is holomorphic on $D(0, R_+) = D(0, R)$ and g_- is holomorphic on $D(0, R_-)$. So f_- is holomorphic on $\{|z| > 1/R_-\} = \{|z| > r\}$, and $f = f_+ + f_-$ is holomorphic on A.

Moreover, we have $f_+'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ and $g_-'(w) = \sum_{n=1}^{\infty} n a_{-n} w^{n-1}$. Using chain rule, we get

$$f_-'(z) = g_-'(\frac{1}{z}) \cdot \frac{-1}{z^2} = \sum_{n=1}^{\infty} -n a_{-n} z^{-n-1} = \sum_{k=-\infty}^{-1} k a_k z^{k-1}.$$

Thus, the derivative of $f = f_+ + f_-$ is

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n z^{n-1}.$$

Theorem 4.1.1. *Let $r < R \in [0, \infty]$. Suppose that f is holomorphic on $A = \{z : r < |z| < R\}$. Then f has a Laurent expansion:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad z \in A,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z|=t} \frac{f(z)}{z^{n+1}}\, dz, \quad n \in \mathbb{Z}, r < t < R. \tag{4.1}$$

In the proof we will use the Laurent series expansion of a particular function $f(z) = \frac{1}{z-z_0}$, where $z_0 \in \mathbb{C} \setminus \{0\}$ is fixed. Let $r = |z_0|$. Note that f is holomorphic on the disc $\{|z| < r\}$ and the annulus $\{r < |z| < \infty\}$. In the disc, we have $|z/z_0| < 1$, so

$$f(z) = \frac{-z_0}{1 - z/z_0} = -z_0 \sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n = \sum_{n=0}^{\infty} \frac{-z^n}{z_0^{n+1}}.$$

This means that $a_n = 0$ if $n < 0$; $a_n = -1/z_0^{n+1}$ if $n \geq 0$. In the annulus, we have $|z_0/z| > 1$, so

$$f(z) = \frac{1/z}{1 - z_0/z} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{z_0}{z}\right)^n = \sum_{n=0}^{\infty} \frac{z_0^n}{z^{n+1}} = \sum_{m=-\infty}^{-1} \frac{z^m}{z_0^{m+1}}.$$

That is, $a_n = 0$ if $n \geq 0$, $a_n = 1/z_0^{n+1}$ if $n < 0$.

The above method can be used to derive the Laurent series of more complicated functions. For example, $f(z) = \frac{1}{(z-1)(z-2)}$. Note that, we can write $f(z) = \frac{1}{z-2} - \frac{1}{z-1}$. We have the Laurent series expansions of $\frac{1}{z-1}$ in the two regions: $\{|z| < 1\}$ and $\{|z| > 1\}$, and the Laurent series expansions of $\frac{1}{z-2}$ in the two regions: $\{|z| < 2\}$ and $\{|z| > 2\}$. Putting them together, we can then derive the Laurent series of f in the regions: $\{|z| < 1\}$, $\{1 < |z| < 2\}$, and $\{|z| > 2\}$, respectively.

Proof. First, from Cauchy's theorem, the value of each a_n does not depend on t. Let $z_0 \in A$. Pick $s < S \in (r, R)$ such that $s < |z_0| < S$. Let $\varepsilon = \min\{|z_0| - s, S - |z_0|\}/2 > 0$. Let $J_1 = \{|z| = S\}$, $J_2 = \{|z| = s\}$, and $J_3 = \{|z - z_0| = \varepsilon\}$. Then J_2 and J_3 lie inside J_1. The function $\frac{f(z)}{z-z_0}$ is holomorphic on J_1, J_2, J_3, and the domain bounded by these circles. From Cauchy's Theorem and Cauchy's formula,

$$\int_{J_1} \frac{f(z)}{z - z_0}\, dz - \int_{J_2} \frac{f(z)}{z - z_0}\, dz = \int_{J_3} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

Now we expand $\frac{1}{z-z_0}$ using

$$\frac{1}{z-z_0} = \frac{1/z}{1-z_0/z} = \sum_{n=0}^{\infty} \frac{z_0^n}{z^{n+1}}, \quad z \in J_1.$$

$$\frac{1}{z-z_0} = \frac{-1/z_0}{1-z/z_0} = \sum_{k=0}^{\infty} \frac{-z^k}{z_0^{k+1}} = \sum_{k=-\infty}^{-1} \frac{-z_0^n}{z^{n+1}}, \quad z \in J_2.$$

The first holds because $|z_0/z| < 1$ for $z \in J_1$. The second holds because $|z/z_0| < 1$ for $z \in J_2$. Thus,

$$\begin{aligned} 2\pi i f(z_0) &= \int_{J_1} \frac{f(z)}{z-z_0}\, dz - \int_{J_2} \frac{f(z)}{z-z_0}\, dz \\ &= \int_{J_1} \sum_{n=0}^{\infty} \frac{f(z) z_0^n}{z^{n+1}}\, dz + \int_{J_2} \sum_{n=-\infty}^{-1} \frac{f(z) z_0^n}{z^{n+1}}\, dz. \end{aligned}$$

If the infinite sums exchange with the integrals, we have

$$2\pi i f(z_0) = \sum_{n=0}^{\infty} \Big(\int_{J_1} \frac{f(z)}{z^{n+1}}\, dz \Big) z_0^n + \sum_{n=-\infty}^{-1} \Big(\int_{J_2} \frac{f(z)}{z^{n+1}}\, dz \Big) z_0^n = \sum_{n=-\infty}^{\infty} 2\pi i a_n z_0^n. \tag{4.2}$$

It remains to show that the two series inside the integrals converge uniformly on the curves. Note that, for $z \in J_1$,

$$\Big| \frac{f(z) z_0^n}{z^{n+1}} \Big| \le \|f\|_{J_1} \frac{|z_0|^n}{R^{n+1}},$$

and from $|z_0|/R < 1$, we find that

$$\sum_{n=0}^{\infty} \|f\|_{J_1} \frac{|z_0|^n}{R^{n+1}} = \frac{\|f\|_{J_1}}{R} \sum_{n=0}^{\infty} \Big(\frac{|z_0|}{R} \Big)^n < \infty.$$

From comparison principle, we see that $\sum_{n=0}^{\infty} \frac{f(z) z_0^n}{z^{n+1}}$ converges uniformly over $z \in J_1$. For $z \in J_2$,

$$\Big| \frac{f(z) z_0^n}{z^{n+1}} \Big| \le \|f\|_{J_2} \frac{|z_0|^n}{r^{n+1}},$$

and from $|z_0|/r > 1$, we find that

$$\sum_{n=-\infty}^{-1} \|f\|_{J_2} \frac{|z_0|^n}{r^{n+1}} = \frac{\|f\|_{J_2}}{r} \sum_{k=1}^{\infty} \Big(\frac{r}{|z_0|} \Big)^k < \infty.$$

From comparison principle, we see that $\sum_{n=-\infty}^{-1} \frac{f(z) z_0^n}{z^{n+1}}$ converges uniformly over $z \in J_2$. The proof is now finished. $\square$

Remark. We will not use the above theorem to calculate the coefficients a_n. Instead, we will find the a_n using other methods, and then use this theorem to calculate the value of integrals. We will prove in a homework problem that the Laurent series expansion is unique. So we may use the known Laurent series to compute the integrals $\int_{|z|=t} \frac{f(z)}{z^{n+1}}\,dz$ for $n \in \mathbb{Z}$.

For example, the Laurent series expansion of $e^{1/z}$ is

$$\sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{-\infty}^{0} \frac{z^n}{(-n)!}.$$

So $a_n = 0$ if $n > 0$ and $a_n = 1/(-n)!$ if $n \le 0$. A similar example is $e^{-1/z^2} = \sum_{n=0}^{\infty} (-1/z^2)^n n!$. Then we have

$$\int_{|z|=1} e^{1/z} dz = 2\pi i a_{-1} = 2\pi i.$$

Similarly, if f is holomorphic on $A = \{r < |z - z_0| < R\}$, then f has a unique Laurent series expansion in A:

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=t} \frac{f(z)}{(z - z_0)^{n+1}}\,dz, \quad n \in \mathbb{Z}, r < t < R.$$

Homework. Chapter V §2: 8
Additional:

1. Suppose that f is holomorphic on $A = \{r < |z| < R\}$, where $0 \le r < R \le \infty$. Suppose that there are two series of complex numbers $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{Z}}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-\infty}^{\infty} b_n z^n, \quad z \in A.$$

   Show that $a_n = b_n$ for all $n \in \mathbb{Z}$. This means that the Laurent series expansion is unique. Hint: It suffices to show that if $f \equiv 0$, then $a_n = 0$ for all n. Use $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=-\infty}^{-1} -a_n z^n$ to construct a bounded entire function.

2. Suppose f is holomorphic on $\{r < |z| < R\}$. Prove that for any $s \in (r, R)$,

$$\int_{|z|=s} \frac{f'(z)}{z^n}\,dz = n \int_{|z|=s} \frac{f(z)}{z^{n+1}}\,dz, \quad n \in \mathbb{Z}.$$

4.2 Isolated Singularities

Suppose f is holomorphic on U, $z_0 \notin U$, but there is $r > 0$ such that $D(z_0, r) \setminus \{z_0\} \subset U$. Then we say that z_0 is an isolated singularity of f. Then f has a Laurent expansion in $\{0 < |z - z_0| < r\}$:

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{4.3}$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=t} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \quad n \in \mathbb{Z}, \quad t \in (0, r). \tag{4.4}$$

Definition 4.2.1. *The series $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ is called the principal part at z_0 of f.*

Case 1: $a_{-n} = 0$ for all $n \in \mathbb{N}$, i.e., the principal part vanishes. Then (4.3) becomes the usual power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, which converges to a holomorphic function in $\{|z - z_0| < r\}$. Thus, if we define $f(z_0) = a_0$, then f is holomorphic on $U \cup \{z_0\}$. In this case, we call z_0 a removable singularity.

Case 2: Not all a_{-n}, $n \in \mathbb{N}$, equal to 0, and there are only finitely many nonzero a_{-n}. We may find $m \in \mathbb{N}$ such that $a_{-m} \neq 0$ and $a_{-n} = 0$ for $n > m$. In this case, we call z_0 a pole of f of order m. We find that z_0 is a removable singularity of $g(z) := (z - z_0)^m f(z)$, and $g(z_0) = a_{-m} \neq 0$. A pole of order 1 is called a simple pole.

Case 3: There are infinitely many nonzero a_{-n}, $n \in \mathbb{N}$. In this case, we call z_0 an essential singularity of f. For any $m \in \mathbb{N}$, z_0 is still a (essential) singularity of $(z - z_0)^m f(z)$.

Examples.

1. Suppose f is holomorphic on an open set U, and $z_0 \in U$. Now we remove the definition of f at z_0. Then z_0 becomes a removable singularity of f.

2. 0 is pole of order 1 of $f(z) = \frac{1}{z}$. In fact, $\frac{1}{z}$ is already a Laurent series of f at 0.

3. Since the Laurent series expansion of $e^{1/z}$ at 0 is $\sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}$, there are infinitely many $n < 0$ such that $a_n \neq 0$. So 0 is an essential singularity of $e^{1/z}$.

Suppose there is $m \in \mathbb{Z}$ such that $a_m \neq 0$ and for all $n < m$, $a_n = 0$. This means that z_0 is either a removable singularity or a pole, and f is not constant 0 near z_0. In this case, we say that the order of f at z_0 is m, and write

$$\operatorname{ord}_{z_0} f = m.$$

We see that $\operatorname{ord}_{z_0} f = m$ if and only if there is a holomorphic function g in $D(z_0, r)$ with $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^m g(z)$. If $m \geq 0$, z_0 is removable. If $m \geq 1$, z_0 is a zero of f after removing the singularity, and we say that z_0 is a zero of f of order m. A zero of order 1 is called a simple zero. Since $a_n = \frac{f^{(n)}(z_0)}{n!}$, z_0 is a zero of order m iff $f^{(k)}(z_0) = 0$ for $0 \le k \le m-1$ and $f^{(m)}(z_0) \ne 0$. If $m < 0$, z_0 is a pole of f of order $|m|$.

Note that if f and g are holomorphic at z_0, and if $f(z_0), g(z_0) \ne 0$, then $h_1 = fg$ and $h_2 = f/g$ are both holomorphic at z_0, and $h_1(z_0), h_2(z_0) \ne 0$. This means that $\operatorname{ord}_{z_0} f = \operatorname{ord}_{z_0} g = 0$ implies that $\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0}(f/g) = 0$. Now if $\operatorname{ord}_{z_0} f = m$ and $\operatorname{ord}_{z_0} g = n$, then there are F and G, which are holomorphic at z_0 with $F(z_0), G(z_0) \ne 0$, such that $f(z) = (z-z_0)^m F(z)$ and $g(z) = (z-z_0)^n G(z)$. Then we get

$$f(z)g(z) = (z-z_0)^{m+n} F(z)G(z), \quad f(z)/g(z) = (z-z_0)^{m-n} F(z)/G(z).$$

Thus, we have

$$\operatorname{ord}_{z_0}(f \cdot g) = \operatorname{ord}_{z_0} f + \operatorname{ord}_{z_0} g, \quad \operatorname{ord}_{z_0}(f/g) = \operatorname{ord}_{z_0} f - \operatorname{ord}_{z_0} g.$$

Examples.

1. We have $\operatorname{ord}_{z_0} 1 = 0$ for any $z_0 \in \mathbb{C}$, $\operatorname{ord}_0 z = \operatorname{ord}_0 \sin z = 1$ (because the derivative of z and $\sin z$ does not vanish at 0). Thus, $\operatorname{ord}_0 1/z = \operatorname{ord}_0 1/\sin z = -1$, which implies that 0 is a simple pole of $1/z$ and $1/\sin z$. From $\operatorname{ord}_0 \sin z/z = \operatorname{ord}_0 \sin z - \operatorname{ord}_0 z = 0$, we see that 0 is a removable singularity of $\sin z/z$. After removing the singularity 0, we extend $\sin z/z$ to an entire function.

Definition 4.2.2. *Let U be an open set. Suppose that S is a relatively closed subset of U. If f is holomorphic on $U \setminus S$, and each $z_0 \in S$ is a pole of f, then we say that f is meromorphic on U.*

Suppose f and g are holomorphic on a domain U such that g is not constant 0. Then f/g is meromorphic on U after removing those removable singularities.

The quotient of two entire functions is meromorphic on $\mathbb{C}$. This includes the quotient of two polynomials, which is called a rational function. The functions $\tan z = \frac{\sin z}{\cos z}$ and $\cot z = \frac{\cos z}{\sin z}$ are also meromorphic in $\mathbb{C}$. For $\tan z$, since the zeroes of $\cos z$ are $k\pi + \pi/2$, $k \in \mathbb{Z}$, which are simple because $\cos' z = -\sin z \ne 0$ at $k\pi + \pi/2$, and since $\sin(k\pi + \pi/2) \ne 0$, we find that every $k\pi + \pi/2$ is a simple pole of $\tan z$. Similarly, $\cot z$ is also a meromorphic function in $\mathbb{C}$, whose poles are $k\pi$, $k \in \mathbb{Z}$, and every pole is simple.

Now we describe the behavior of f near an isolated singularity of each kind. We will always assume that z_0 is a singularity of f, and f is holomorphic on $D(z_0, r) \setminus \{z_0\}$.

Theorem 4.2.1. *The following are equivalent*

(i) *z_0 is a removable singularity of f;*

(ii) $\lim_{z \to z_0} f(z)$ *converges;*

(iii) *there is $r' \in (0, r)$ such that f is bounded in $D(z_0, r') \setminus \{z_0\}$.*

Proof. That (i) implies (ii) is obvious because after the analytic extension, $\lim_{z\to z_0} f(z) = f(z_0) \in \mathbb{C}$. Suppose (ii) holds, and let $w_0 = \lim_{z\to z_0} f(z) \in \mathbb{C}$. Then there is $r' \in (0,r)$ such that $|f(z) - w_0| < 1$ for $z \in D(z_0, r')$, which implies that $|f(z)| \le |w_0| + 1$ in $D(z_0, r') \setminus \{z_0\}$. So we get (iii)

Finally, we show that (iii) implies (i). Suppose $|f(z)| \le M < \infty$ on $D(z_0, r') \setminus \{z_0\}$. From (4.4), we see that, for any $t \in (0, r')$,

$$|a_n| \le \frac{1}{2\pi} M t^{-n-1} L(\{|z - z_0| = t\}) \le M t^{-n}, \quad n \in \mathbb{Z}.$$

If $n \le -1$, then $t^{-n} \to 0$ as $t \to 0$, which implies that $a_n = 0$ for $n \le -1$. So we get (i). $\square$

Theorem 4.2.2. *z_0 is a pole of f $\Leftrightarrow$ $\lim_{z\to z_0} |f(z)| = \infty$.*

Proof. z_0 is a pole of f $\Leftrightarrow$ z_0 is a zero of $1/f$ $\Leftrightarrow$ $\lim_{z\to z_0} |1/f(z)| = 0 \Leftrightarrow \lim_{z\to z_0} |f(z)| = \infty$. Here that $\lim_{z\to z_0} |1/f(z)| = 0$ implies z_0 is a zero of $1/f$ follows from the above theorem: we first conclude that z_0 is a removable singularity of $1/f$ using the boundedness of $1/f$ near z_0, and then use the limit to see that the extended value of $1/f$ at z_0 is 0. $\square$

Recall that $S \subset \mathbb{C}$ is dense in $\mathbb{C}$ if $\overline{S} = \mathbb{C}$, which is equivalent to the following: for any $w_0 \in \mathbb{C}$ and $r > 0$, $D(w_0, r) \cap S \ne \emptyset$.

Theorem 4.2.3. *z_0 is an essential singularity of f $\Leftrightarrow$ for any $t \in (0,r)$, $f(D(z_0,t) \setminus \{z_0\})$ is dense in $\mathbb{C}$.*

Proof. We first prove the $\Leftarrow$ part. Assume that $f(D(z_0,t) \setminus \{z_0\})$ is dense in $\mathbb{C}$ for any $t \in (0,r)$. If z_0 is a removable singularity, then $\lim_{z\to z_0} f(z)$ exists. So there is $t \in (0,r)$ such that $f(D(z_0,t) \setminus \{z_0\})$ is contained in a disc, so it can not be dense in $\mathbb{C}$. If z_0 is a pole, then $\lim_{z\to z_0} |f(z)| = \infty$. Then there is $t \in (0,r)$ such that $f(D(z_0,t) \setminus \{z_0\}) \subset \{|z| > 1\}$, which also can not be dense in $\mathbb{C}$. So z_0 must be an essential singularity.

Then we prove the $\Rightarrow$ part. Assume that z_0 is an essential singularity, but $f(D(z_0,t) \setminus \{z_0\})$ is not dense in $\mathbb{C}$ for some $t \in (0,r)$. Then there exist $w_0 \in \mathbb{C}$ and $\varepsilon > 0$ such that $|f(z) - w_0| \ge r$ for every $z \in D(z_0,t) \setminus \{z_0\}$. Let $g(z) = \frac{1}{f(z) - w_0}$. Then g is holomorphic and bounded in U. So z_0 is a removable singularity of g. Since $f(z) = w_0 + \frac{1}{g(z)}$ for $z \in U$, we see that z_0 is either a removable singularity (if $g(z_0) \ne 0$) or a pole (if $g(z_0) = 0$) of f, which is a contradiction. $\square$

Remark. Actually, it is known that the $f(D(z_0,t) \setminus \{z_0\})$ in the above theorem is either the whole $\mathbb{C}$ or $\mathbb{C}$ without a single point. Using a homework problem, one can show that, if $f(z) = e^{1/z}$, then for any $r > 0$, $f(D(0,r) \setminus \{0\}) = \mathbb{C} \setminus \{0\}$.

Homework

Let f be an entire function that satisfies $\lim_{|z|\to\infty} |f(z)| = \infty$. Prove that f is a polynomial. Hint: Consider $g(z) := f(1/z)$.

4.3 The Residue Formula

Let z_0 be an isolated singularity of f, and let $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be the Laurent series of f in $0 < |z - z_0| < r$ for some $r > 0$. We call a_{-1} the residue of f at z_0, and write

$$\operatorname{Res}_{z_0} f = a_{-1}.$$

Lemma 4.3.1. *For any $t \in (0, r)$. Then*

$$\int_{\{|z-z_0|=t\}} f = 2\pi i a_{-1} = 2\pi i \operatorname{Res}_{z_0} f.$$

Proof. Recall that for any $n \in \mathbb{Z}$,

$$a_n = \frac{1}{2\pi i} \int_{\{|z-z_0|=t\}} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Taking $n = -1$, we get the desired equality. □

Theorem 4.3.1. *f has a primitive in $D(z_0, r) \setminus \{z_0\}$ iff $\operatorname{Res}_{z_0} f = 0$.*

Proof. If f has a primitive in $D(z_0, r) \setminus \{z_0\}$, then $\int_C f = 0$ for $C = \{|z-z_0| = t\}$, which implies that $a_{-1} = 0$. If $a_{-1} = 0$, then a primitive of f can be represented by

$$\sum_{n=-\infty}^{-2} \frac{a_n}{n+1}(z-z_0)^{n+1} + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}.$$

□

Theorem 4.3.2. [Residue Formula for Jordan Curves] *Let J be a positively oriented Jordan curve. Suppose that f is holomorphic on $\operatorname{Int}(J) \cup J$ except at a finite number of points $z_1, \ldots, z_n \in \operatorname{Int}(J)$. Then*

$$\int_J f = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z_j} f.$$

Proof. Choose $r > 0$ such that the closed discs $\overline{D}(z_k, r)$, $1 \le k \le n$, are mutually disjoint, and all contained in the interior of J. From Cauchy's Theorem and the previous lemma, we get

$$\int_J f = \sum_{k=1}^{n} \int_{|z-z_k|=r} f = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f.$$

□

Remark. The Cauchy's theorem and Cauchy's formula can be viewed as special cases of Residue formula. Take Cauchy's formula for example. We have

$$\int_J \frac{f(z)}{(z-w)^{m+1}}\,dz = 2\pi i \operatorname{Res}_w \frac{f(z)}{(z-w)^{m+1}}.$$

Now we calculate the residue of $\frac{f(z)}{(z-w)^{n+1}}$ at w. Recall that f has a power series expansion near w:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^n, \quad |z-w| < r.$$

Thus,

$$\frac{f(z)}{(z-w)^{m+1}} = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z-w)^{n-m-1}, \quad 0 < |z-w| < r.$$

For the residue, we look for the coefficients that corresponds to the term $(z-w)^{-1}$, i.e., when $n = m$. So we get

$$\operatorname{Res}_w \frac{f(z)}{(z-w)^{m+1}} = \frac{f^{(m)}(w)}{m!}.$$

Thus,

$$\int_J \frac{f(z)}{(z-w)^{m+1}}\,dz = 2\pi i \frac{f^{(m)}(w)}{m!}.$$

Moving $2\pi i$ and $m!$ to the left, we get the Cauchy's formula.

To apply the Residue Formula, we need to know how to find residues. The most general method is to find the Laurent series. As we have seen above, if f is holomorphic at z_0, and $g(z) = \frac{f(z)}{(z-z_0)^m}$, then we may use the power series expansion of f to find the Laurent series of g at z_0, and so find $\operatorname{Res}_{z_0} g$. In fact, if $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ near z_0, then $g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n-m}$ near z_0, and $\operatorname{Res}_{z_0} g = a_{m-1}$ because $n = m-1$ corresponds to $(z-z_0)^{-1}$ in the expansion of g.

Examples.

1. Find the residue of $\frac{\sin z}{z^6}$ at 0. If $\sin z = \sum_{n=0}^{\infty} a_n z^n$, then $\operatorname{Res}_0 \frac{\sin z}{z^6} = a_5$. We may compute a_5 by $a_5 = \frac{\sin^{(5)}(0)}{5!} = \frac{\sin'(0)}{120} = \frac{1}{120}$. So the residue is $\frac{1}{120}$.

2. Find the residue of $f(z) = \frac{z^2}{(z+1)(z-1)^2}$ at 1. Let $g(z) = \frac{z^2}{z+1}$. Then g is holomorphic at 1, and $f(z) = \frac{g(z)}{(z-1)^2}$. If the power series of g at 1 is

$$\sum_{n=0}^{\infty} a_n (z-1)^n,$$

then the Laurent series of f at 1 is

$$\frac{a_0}{z^2} + \frac{a_1}{z} + \text{higher terms}.$$

So the residue is a_1. The value of a_1 can be computed by $a_1 = g'(1)$. Since

$$g'(z) = \frac{2z}{z+1} - \frac{z^2}{(z+1)^2},$$

we have $a_1 = g'(1) = \frac{3}{4}$. So the residue is $\frac{3}{4}$.

3. Let $f(z) = \frac{z^2}{(z+1)(z-1)^2}$. Find $\int_{|z-1|=1} f$. Note that f is meromorphic in $\mathbb{C}$ with two poles -1 and 1. Since -1 lies outside $\{|z-1|=1\}$, and 1 lies inside $\{|z-1|=1\}$, by Residue formula, we have

$$\int_{|z-1|=1} f = 2\pi i \operatorname{Res}_1 f = 2\pi i \cdot \frac{3}{4}.$$

Lemma 4.3.2. *Let f and g be holomorphic at z_0. Suppose $f(z_0) = 0$ and $f'(z_0) \neq 0$, i.e.,* $\operatorname{ord}_{z_0} f = 1$. *Then*

$$\operatorname{Res}_{z_0}\left(\frac{g}{f}\right) = \frac{g(z_0)}{f'(z_0)}.$$

Proof. We may write $f(z) = F(z)(z-z_0)$, where F is holomorphic at z_0, and $F(z_0) = f'(z_0) \neq 0$. Then

$$\frac{g(z)}{f(z)} = \frac{1}{z-z_0}\frac{g(z)}{F(z)}.$$

Since g and F are both holomorphic at z_0, and $F(z_0) \neq 0$, we see that $\frac{g}{F}$ is holomorphic at z_0. From the above displaced formula, we conclude that

$$\operatorname{Res}_{z_0}\left(\frac{g}{f}\right) = \frac{g(z_0)}{F(z_0)} = \frac{g(z_0)}{f'(z_0)}.$$

□

Examples.

1. Find the residue of $\cot z$ at $k\pi$, $k \in \mathbb{Z}$.

   We have $\cot z = \frac{\cos z}{\sin z}$. Since $\sin(k\pi) = 0$ and $\sin'(k\pi) = \cos(k\pi) \neq 0$, $k\pi$ is a simple zero of $\sin z$. From the above lemma,

$$\operatorname{Res}_{k\pi} \cot z = \frac{\cos(k\pi)}{\sin'(k\pi)} = \frac{\cos(k\pi)}{\cos(k\pi)} = 1.$$

   Now we can compute $\int_{|z|=5} \cot z\, dz$.

It is easy to see that the poles of $\cot z$ that lie inside $|z| = 5$ are 0, π, and $-\pi$. From Residue Formula,

$$\int_{|z|=5} \cot z\, dz = 2\pi i(\operatorname{Res}_0 \cot z + \operatorname{Res}_\pi \cot z + \operatorname{Res}_{-\pi} \cot z) = 6\pi i.$$

2. Find the residue of $\frac{e^z}{\sin z}$ at 0.

   Since $\sin(0) = 0$ and $\sin'(0) = \cos(0) = 1 \neq 0$, 0 is a simple zero of $\sin z$. From part (b), we get $\operatorname{Res}_0 \frac{e^z}{\sin z} = \frac{e^0}{\sin'(0)} = 1$.

3. Let $f(z) = z^2 - 2z + 3$. Let $R = \{x + iy : -1 \le x \le 3, -2 \le y \le 2\}$ and $C = \partial R$ with positive orientation. Find $\int_C \frac{1}{f}$.

   Since $f(z) = (z-1)^2 + 2$, we see that f has two zeros $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$. Since $f'(z) = 2z - 2$, we see that $f'(1 \pm i\sqrt{2}) \neq 0$. So $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$ are simple zeros of f. Applying the lemma, we get

$$\operatorname{Res}_{1+i\sqrt{2}} \frac{1}{f} = \frac{1}{f'(1+i\sqrt{2})} = \frac{1}{i2\sqrt{2}}, \quad \operatorname{Res}_{1-i\sqrt{2}} \frac{1}{f} = \frac{1}{f'(1-i\sqrt{2})} = \frac{1}{-i2\sqrt{2}}.$$

   Since $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$ both lie inside C, we get

$$\int_C \frac{1}{f} = 2\pi i(\operatorname{Res}_{1+i\sqrt{2}} \frac{1}{f} + \operatorname{Res}_{1-i\sqrt{2}} \frac{1}{f}) = 2\pi i(\frac{1}{i2\sqrt{2}} + \frac{1}{-i2\sqrt{2}}) = 0.$$

Theorem 4.3.3. [The General Residue Formula] *Let γ be a contour in a domain U such that $W(\gamma, \alpha) = 0$ for every $\alpha \in \mathbb{C} \setminus U$. Suppose that f is holomorphic on U except at a set S, which has no accumulation point in U, and does not intersect γ. Then*

$$\int_\gamma f = 2\pi i \sum_{w \in S} W(\gamma, w) \operatorname{Res}_w f.$$

Proof. Let S' denote the set of $w \in S$ that does NOT lie on the unbounded component of $\mathbb{C} \setminus \gamma$. Since S has no accumulation point in U, S' is finite. Note that $W(\gamma, w) = 0$ for $w \in S \setminus S'$. So it suffices to prove the displayed formula with S' in place of S. We may then consider the contour $\eta = \gamma - \sum_{w \in S'} W(\gamma, z)\{|z - w| = r\}$, where $r > 0$ is small such that $\overline{D}(w, r)$, $w \in S'$, are contained in U and mutually disjoint. From the general Cauchy's Theorem, we have $\int_\eta f = 0$. The proof is finished by applying the above residue formula. $\square$

Homework Chapter VI §1: 12, 14, 15, 18, 20, 26(a,d)

4.4 Rouché's Theorem

Suppose $\operatorname{ord}_{z_0} f = m$, i.e., $f(z) = (z - z_0)^m g(z)$, where g is holomorphic at z_0 and $g(z_0) \neq 0$. Then

$$f'(z) = m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z),$$

which implies that

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Since g' and g are holomorphic at z_0, and $g(z_0) \neq 0$, we see that

$$\operatorname{Res}_{z_0} \frac{f'}{f} = \operatorname{ord}_{z_0} f \tag{4.5}$$

.

Theorem 4.4.1. *Let f be meromorphic on U. Let γ be a positively oriented Jordan curve in U such that γ does not pass through any zero or pole of f, and the interior of γ is contained in U. Then we have*

$$\int_\gamma \frac{f'}{f} = 2\pi i \sum_{j=1}^{n} \operatorname{ord}_{z_j} f,$$

where $z_1, \dots, z_n$ are the zeros or poles of f that lie inside γ.

Proof. We know that f'/f is meromorphic on U, whose pole is either a zero or a pole of f. The conclusion follows from (4.5) and the Residue Formula. $\square$

If z_0 is a zero of order m of f, we now say that f has m zeros at z_0 counting multiplicities. If z_0 is a pole of order m of f, we now say that f has m poles at z_0 counting multiplicities. The total number of zeros of f that lie inside γ (c.m.) is the sum of $\operatorname{ord}_{z_j} f$ over those zeros of f inside C. The total number of poles of f that lie inside γ (c.m.) is the sum of $-\operatorname{ord}_{z_j} f$ over those poles of f inside C. The conclusion of the above theorem can be written as

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \#\{\text{zeros of } f \text{ inside } \gamma\} - \#\{\text{poles of } f \text{ inside } \gamma\}.$$

In particular, if f is holomorphic on $\operatorname{Int}(\gamma) \cup \gamma$ such that no zeros of f lie no γ, then $\frac{1}{2\pi i} \int_\gamma f$ is equal to the number of zeros of f (c.m.) inside γ. If the value of the integral is 0, then f has no zeros inside γ. If the value is positive, then f has zeros inside γ. If the value is 1, then f has exactly one zero, which is simple, inside γ.

Theorem 4.4.2. [Rouché's Theorem] *Let J be a Jordan curve. Let f and g be analytic on J and its interior. Suppose that*

$$|f(z) - g(z)| < |f(z)|, \quad z \in J.$$

Then f and g have the same number of zeros (c.m.) inside J.

Proof. It suffices to show that $\int_J \frac{g'}{g} = \int_J \frac{f'}{f}$. The idea is to change continuously from f to g. Let $h = g - f$ and $g_t = f + th$, $0 \le t \le 1$. Then $g_0 = f$ and $g_1 = g$. Since $|h| < |f|$ on J, we see that $g_t \ne 0$ on J for $0 \le t \le 1$. Let

$$m(t) = \frac{1}{2\pi i}\int_J \frac{g_t'}{g_t} = \frac{1}{2\pi i}\int_J \frac{f' + th'}{f + th}, \quad 0 \le t \le 1.$$

Then $m(t)$ equals to the number of zeros (c.m.) of g_t inside J. In particular, $m(0)$ is the number of zeros of f inside J, and $m(1)$ is the number of zeros of g inside J. We see that $m(t) \in \mathbb{Z}$ for $0 \le t \le 1$. One may also show that $m(t)$ is continuous in t. So $m(t)$ has to be a constant. Then $m(0) = m(1)$, as desired. □

Example. Let $P(z) = z^8 - 5z^3 + z - 2$. We want to find the number of zeros (c.m.) of P inside $\{|z| < 1\}$. We compare it with $f(z) = -5z^3$. Then on $\{|z| = 1\}$,

$$|P(z) - f(z)| = |z^8 + z - 2| \le |z|^8 + |z| + |2| = 1 + 1 + 2 = 4 < 5 = |-5z^3| = |f(z)|.$$

From Rouché's Theorem, we see that P and f have the same number of zeros inside $\{|z| < 1\}$. Since 0 is the only zero of f, which has order 3, we conclude that P has 3 zeros (c.m.) in $\{|z| < 1\}$.

Next, we want to find the number of zeros (c.m.) of P inside $\{|z| < 2\}$. We compare it with $f(z) = z^8$. Then on $\{|z| = 2\}$,

$$|P(z) - f(z)| = |-5z^3 + z - 2| \le 5|z|^3 + |z| + |2| = 5 * 2^3 + 2 + 2 = 44 < 2^8 = |z^8| = |f(z)|.$$

From Rouché's Theorem, we see that P and f have the same number of zeros inside $\{|z| < 2\}$. Since 0 is the only zero of f, which has order 8, we conclude that P has 8 zeros (c.m.) in $\{|z| < 2\}$.

Combining the above two results, we can conclude that P has 5 zeros (c.m.) in $\{1 \le |z| < 2\}$.

For polynomials, we have a method to determine whether all of its zeros are simple. In fact, if P has a multiple zero at z_0, then z_0 is also a zero of P'. This means that the greatest common divisor (P, P'), which is also a polynomial, has a zero at z_0. Thus, if (P, P') has a low degree, then we may find all multiple zeros of P. For example, if $P(z) = z^8 - 5z^3 + z - 2$, then $P'(z) = 8z^7 - 15z^2 + 1$. On can calculate that $(P, P') = 1$, which has no zero. So all zeros of P are simple zeros. The conclusions in the previous paragraph hold without counting multiplicities.

Example. We may use Rouché's theorem to get another proof of Fundamental Theorem of Algebra. Let $P(z) = \sum_{k=0}^n a_k z^k$ be a polynomial of degree n. So $a_n \ne 0$. Compare $P(z)$ with $f(z) = a_n z^n$. The only zero of f is 0, which has order n. We have

$$\frac{|P(z) - f(z)|}{|f(z)|} \le \sum_{k=0}^{n-1} \frac{|a_k|}{|a_n|}|z|^{k-n} \to 0, \quad \text{as} |z| \to \infty.$$

So we may find $N > 0$ such that, if $|z| > N$, then $\frac{|P(z)-f(z)|}{|f(z)|} < 1$, which implies that $|P(z) - f(z)| < |f(z)|$ on $\{|z| = R\}$ for any $R > N$. From Rouché's theorem, we conclude that P has n zeros (c.m.) in $\{|z| < R\}$ if $R > N$. Thus, P has n zeros (c.m.) in $\mathbb{C}$.

Homework Chapter VI §1: 31, 32, 35.

4.5 The Open and Inverse Mapping Theorem

Theorem 4.5.1. [Open Mapping Theorem] *Let f be analytic in an open set U such that f is not constant in any open disc. Then $f(U) = \{f(z) : z \in U\}$ is open.*

Proof. Let $w_0 \in f(U)$, we will show that there is $\varepsilon > 0$ such that $D(w_0, \varepsilon) \subset f(U)$. Let $z_0 \in U$ be such that $f(z_0) = w_0$. Suppose that the power series of f at z_0 is

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Then $a_0 = w_0$. Now z_0 is a zero of $f - w_0$. Let $m = \mathrm{ord}_{z_0}(f - w_0) \in \mathbb{N}$, which is well defined since f is not constant near z_0. Then $a_m \neq 0$ and $a_n = 0$ for $1 \le n \le m-1$. When z is near z_0,

$$f(z) = w_0 + \sum_{n=m}^{\infty} a_n (z - z_0)^n.$$

Let

$$g(z) = w_0 + a_m (z - z_0)^m;$$

$$h(z) = \frac{f(z) - g(z)}{(z - z_0)^m} = \sum_{n=m+1}^{\infty} a_n (z - z_0)^{n-m}.$$

Then h is analytic at z_0, and $h(0) = 0$. Pick $r > 0$ such that $\overline{D}(z_0, r) \subset U$ and $\|h\|_{|z-z_0|=r} < \frac{|a_m|}{2}$. Let $\varepsilon = \frac{|a_m|}{2} r^m > 0$. Fix any $w \in D(w_0, \varepsilon)$. Let $f_w(z) = f(z) - w$ and $g_w(z) = g(z) - w$. We will apply Rouché's Theorem and show that $w \in f(D(z_0, r))$, which is equivalent to that f_w has a zero in $D(z_0, r)$. Note that $g_w(z) = 0$ iff $(z - z_0)^m = \frac{w - w_0}{a_m}$. Thus, g_w has m zeros (c.m.), whose distance from z_0 are all equal to $(\frac{|w-w_0|}{|a_m|})^{1/m}$, which is less than $(\frac{\varepsilon}{|a_m|})^{1/m} < r$. Thus, g_w has m zeros in $\{|z - z_0| < r\}$.

For $z \in \{|z - z_0| = r\}$,

$$|g_w(z)| = |w_0 + a_m (z - z_0)^m - w| \ge |a_m (z - z_0)^m| - |w_0 - w| > |a_m| r^m - \varepsilon \ge \frac{|a_m| r^m}{2},$$

and

$$|f_w(z) - g_w(z)| = |f(z) - g(z)| = r^m |h(z)| \le r^m \|h\|_{|z-z_0|=r} < \frac{|a_m| r^m}{2} < |g_w(z)|, \quad |z - z_0| = r.$$

From Rouché's theorem, f_w also has m zeros (c.m.) in $D(z_0, r)$. Thus, $w \in f(U)$ for every $w \in D(w_0, \varepsilon)$, i.e., $D(w_0, \varepsilon) \subset f(U)$. $\square$

Remark. If U is a domain, then the condition in the theorem is equivalent to that f is not constant. We have seen before that if a holomorphic function f defined on a domain U satisfies that $|f|$ is constant or $3\operatorname{Re} f + 2\operatorname{Im} f = 0$, then f is constant. Now we see such kind of results all follow easily from the open mapping theorem.

Definition 4.5.1. *An analytic function f defined on U is called an analytic isomorphism if there is an analytic function g defined on $f(U)$ such that $g(f(z)) = z$ for every $z \in U$. If f is an analytic isomorphism defined on U such that $f(U) = U$, then we say that f is an analytic automorphism of U. We say that f is a local analytic isomorphism at a point z_0 if there exists an open set U containing z_0 such that f is an analytic isomorphism on U.*

Homework

1. Suppose f_1 is an analytic isomorphism defined on U, and f_2 is an analytic isomorphism defined on $f_1(U)$. Prove that f_1^{-1} and $f_2 \circ f_1$ are also analytic isomorphisms.

It is clear that an analytic isomorphism f must be injective. The following theorem states that the converse is also true.

Theorem 4.5.2. *Let f be holomorphic and injective on an open set U. Then f is an analytic isomorphism.*

Proof. Define $g = f^{-1}$ on $f(U)$. Since f is injective, g is well defined. It suffices to show that g is holomorphic. Since f is injective, it is not constant in any open disc. From the open mapping theorem, for any open set $O \subset U$, $g^{-1}(O) = f(O)$ is open. This shows that g is continuous.

Let Z denote the set of zeros of f' in U. Then Z has no accumulation point in U. Let $U' = U \setminus Z$. Then U' is also open. Fix $w_0 \in f(U')$. Let $z_0 = g(w_0) \in U'$. Then $w_0 = f(z_0)$. Since g and $g^{-1} = f$ are both continuous, we see that $w \to w_0$ iff $g(w) \to g(w_0)$. Thus,

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{g(w) \to g(w_0)} \frac{g(w) - g(w_0)}{f(g(w)) - f(g(w_0))} = \frac{1}{f'(z_0)},$$

where in the last step we used that $f'(z_0) \neq 0$ as $z_0 \notin Z$. Thus, $g'(w_0)$ exists. So g is holomorphic on $f(U')$.

Now we restrict g to $f(U') = f(U) \setminus f(Z)$. Then every $w_0 \in f(Z)$ is a singularity of $g|_{f(U')}$. Since g is continuous on $f(U)$, for any $w_0 \in f(Z)$, $\lim_{w \to w_0} g(w) = g(w_0)$. Thus, every $w_0 \in f(Z)$ is a removable singularity of $g|_{f(U')}$. If we extend $g|_{f(U')}$ to a holomorphic function on $f(U)$, then we must get the function g, because g is already continuous on $f(U)$. So g is holomorphic on $f(U)$. □

If f and g are both holomorphic, and $g = f^{-1}$, then from Chain rule, $g'(f(z))f'(z) = 1$ for $z \in U$, which implies that $f'(z) \neq 0$ for $z \in U$. Thus, if f is a local analytic isomorphism at z_0, then $f'(z_0) \neq 0$. On the other hand, $f'(z) \neq 0$ for $z \in U$ does not imply that f is an analytic isomorphism. One example is $f(z) = e^z$. Note that $(e^z)' = e^z \neq 0$ for all $z \in \mathbb{C}$, but e^z is not injective on $\mathbb{C}$.

Theorem 4.5.3. **[Inverse Mapping Theorem]** *If f is holomorphic at z_0, and $f'(z_0) \neq 0$, then f is a local analytic isomorphism at z_0.*

Proof. We repeat the proof of the open mapping theorem. Let $w_0 = f(z_0)$ and $m = \operatorname{ord}_{z_0}(f - w_0)$. Since $f'(z_0) \neq 0$, $m = 1$. In that proof, we find $r, \varepsilon > 0$ such that for any $w \in D(w_0, \varepsilon)$, $f_w = f - w$ has m zeros (c.m.) in $D(z_0, r)$. Since $m = 1$, this means that, for every $w \in D(w_0, \varepsilon)$, there is exactly one $z \in D(z_0, r)$ such that $f(z) = w$. Let $U_0 = f^{-1}(D(w_0, \varepsilon)) \cap D(z_0, r)$. Then f is injective on U_0, which is open and contains z_0. From the previous theorem, $f|_{U_0}$ is an analytic isomorphism. $\square$

Homework Chapter V, §3: 7. Additional:

1. Suppose f is analytic on $D(z_0; R) \setminus \{z_0\}$, and z_0 is a pole of f. Prove that for any $r \in (0, R)$, there is $M \in (0, \infty)$ such that $f(D(z_0; r) \setminus \{z_0\}) \supset \{z \in \mathbb{C} : |z| > M\}$.

2. Let f be analytic on a domain U, $z_0 \in U$, and $w_0 = f(z_0)$. Suppose that $\operatorname{ord}_{z_0}(f - w_0) = m \in \mathbb{N}$. Prove that there is an open set U_0 with $z_0 \in U_0 \subset U$ such that $f^{-1}(w_0) \cap U_0 = \{z_0\}$ and $f^{-1}(w) \cap U_0$ contains exactly m elements (without repetition) for $w \in f(U_0) \setminus \{w_0\}$. This means that $f|_{U_0}$ is m-to-1 except at z_0.

4.6 Evaluation of Definite Integrals

We are going to apply the Residue theorem to compute definite integral

$$\int_{-\infty}^{\infty} f(x)dx := \lim_{R\to\infty} \int_{-R}^{0} f(x)dx + \lim_{R\to\infty} \int_{0}^{R} f(x)dx.$$

Suppose that f is holomorphic on the closed upper half plane $\{\operatorname{Im} z \geq 0\}$ except for finitely many isolated singularities: z_k, $1 \leq k \leq n$, in the open upper half plane $\{\operatorname{Im} z > 0\}$. Choose $R > 0$ such that $|z_k| < R$ for $1 \leq k \leq n$. Note that

$$\int_{[-R,R]} f = \int_0^1 f(-R + t(R - (-R)))(R - (-R))dt = \int_{-R}^{R} f(x)dx.$$

Let S_R denote the semicircle with radius R: $\gamma(t) = Re^{it}$, $0 \leq t \leq \pi$. Then $\Gamma := [-R, R] \oplus S_R$ is a positively oriented Jordan curve. From Residue formula, we get

$$\int_{-R}^{R} f(x)dx + \int_{S_R} f = \int_\Gamma f = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f.$$

If we know that $\lim_{R\to\infty} \int_{S_R} f = 0$, then we get

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f. \tag{4.6}$$

The limit $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$ is called the principal value integral, and is denoted by

$$\mathrm{P.V.}\int_{-\infty}^{\infty} f(x)dx.$$

It is slightly different from $\int_{-\infty}^{\infty} f(x)dx$. Their relations are:

1. If $\int_{-\infty}^{\infty} f(x)dx$ exists, then $\mathrm{P.V.}\int_{-\infty}^{\infty} f(x)dx$ exists and is equal to $\int_{-\infty}^{\infty} f(x)dx$.
2. If $\mathrm{P.V.}\int_{-\infty}^{\infty} f(x)dx$ exists, $\int_{-\infty}^{\infty} f(x)dx$ may or may not exist. One counterexample is $f(x) = x$. Since $\int_{-R}^{R} xdx = 0$ for all R, $\mathrm{P.V.}\int_{-\infty}^{\infty} xdx = 0$. But $\int_{-\infty}^{\infty} xdx$ does not exist since $\int_0^{\infty} xdx = +\infty$ and $\int_{-\infty}^{0} xdx = -\infty$.
3. If $f(x)$ is nonnegative or an even function on $\mathbb{R}$, then the existence of $\mathrm{P.V.}\int_{-\infty}^{\infty} f(x)dx$ implies the existence of $\int_{-\infty}^{\infty} f(x)dx$, which must be equal.

Moreover, if f is even, and $\int_{-\infty}^{\infty} f(x)dx$ exists, then

$$\int_0^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx.$$

Example. We want to calculate $\int_{-\infty}^{\infty}\frac{dx}{x^4+1}$. Let $f(z) = \frac{1}{z^4+1}$. Then f is meromorphic in $\mathbb{C}$ with 4 poles: $z_1 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, $z_2 = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, $z_3 = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, $z_4 = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, among which z_1 and z_2 lie in $\{\operatorname{Im} z > 0\}$. Since $(z^4+1)' = 4z^3$ does not equal to 0 at z_j, $1 \le j \le 4$, we get

$$\operatorname{Res}_{z_j} f = \frac{1}{4z_j^3} = \frac{1}{4}\cdot\frac{z_j}{z_j^4} = -\frac{z_j}{4}.$$

To estimate $\int_{S_R} f$, we note that $|f(z)| \le \frac{1}{R^4-1}$ on S_R since $|z^4+1| \ge |z^4| - 1 = R^4 - 1 > 0$ on S_R, which implies that

$$\Big|\int_{S_R} f\Big| \le L(S_R)\|f\|_{S_R} \le \frac{\pi R}{R^4-1} \to 0, \quad R\to\infty.$$

Since f is even, we get

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i(\operatorname{Res}_{z_1} f + \operatorname{Res}_{z_2} f) = -\frac{\pi i}{2}(z_1+z_2) = \frac{\pi}{\sqrt{2}}.$$

Example. Compute $\int_0^{\infty}\frac{\sin x}{x}dx$.

Although we know that $\frac{\sin z}{z}$ extends to an entire function, this does not help use compute the integral. In fact, we will integrate the function $f(z) := \frac{e^{ix}}{x}$. Note that f has exactly one pole in $\mathbb{C}$, which is 0. The pole lies on $\mathbb{R}$, so the above method does not apply. Suppose $R > \varepsilon > 0$. Let S_R be as in the last example. We consider a contour (closed curve):

$$\Gamma = [\varepsilon, R] \oplus S_R \oplus [-R, -\varepsilon] \oplus S_\varepsilon^-.$$

Note that f is holomorphic on and inside Γ. From Cauchy's Theorem,

$$0 = \int_\Gamma f = \int_{S_R} f - \int_{S_\varepsilon} f + \int_\varepsilon^R f(x)dx + \int_{-R}^{-\varepsilon} f(x)dx.$$

Note that

$$\int_\varepsilon^R f(x)dx = \int_\varepsilon^R \frac{\cos x}{x}\,dx + i\int_\varepsilon^R \frac{\sin x}{x}\,dx;$$

$$\int_{-R}^{-\varepsilon} f(x)dx = \int_{-R}^{-\varepsilon} \frac{\cos x}{x}\,dx + i\int_{-R}^{-\varepsilon} \frac{\sin x}{x}\,dx.$$

Since $\frac{\cos x}{x}$ is odd and $\frac{\sin x}{x}$ is even, we get

$$\int_\varepsilon^R f(x)dx + \int_{-R}^{-\varepsilon} f(x)dx = 2i\int_\varepsilon^R \frac{\sin x}{x}\,dx.$$

Thus,

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\Big(\int_\varepsilon^R f(x)dx + \int_{-R}^{-\varepsilon} f(x)dx\Big) = 2i\int_0^\infty \frac{\sin x}{x}\,dx.$$

We will show that $\lim_{R\to\infty}\int_{S_R} f = 0$, and evaluate $\lim_{\varepsilon\to 0}\int_{S_\varepsilon} f$.

First, we have

$$\int_{S_R} f = \int_0^\pi \frac{e^{iRe^{it}}}{Re^{it}} iRe^{it}dt = \int_0^\pi ie^{iRe^{it}}dt.$$

So

$$\Big|\int_{S_R} f\Big| \le \int_0^\pi e^{-R\sin t}dt = 2\int_0^{\pi/2} e^{-R\sin t}dt.$$

The last equality holds because $\sin(\pi - t) = \sin t$.

Lemma 4.6.1. $\sin t \ge \frac{2t}{\pi}$ *for* $0 \le t \le \pi/2$.

Proof. Since $\sin''(t) = -\sin t \le 0$ on $[0, \frac{\pi}{2}]$, $\sin t$ is a concave function on $[0, \frac{\pi}{2}]$. Note that $y = \frac{2t}{\pi}$ is the equation of a straight line that passes through $(0, 0)$ and $(\frac{\pi}{2}, 1)$. Since the curve $y = \sin t$ also passes through these two points, the concaveness of sin on $[0, \frac{\pi}{2}]$ implies that $\sin t \ge \frac{2t}{\pi}$ for $0 \le t \le \pi/2$. $\square$

From the lemma

$$\int_0^{\pi/2} e^{-R\sin t}dt \le \int_0^{\pi/2} e^{-2Rt/\pi}dt \le \int_0^\infty e^{-2Rt/\pi}dt = \frac{\pi}{2R},$$

which tends to 0 as $R \to \infty$.

Now we evaluate $\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \frac{e^{iz}}{z}$. For $\varepsilon > 0$ and $\theta \in (0, 2\pi]$, let $\gamma_{\varepsilon,\theta}$ denote the curve $\gamma_\varepsilon(t) = \varepsilon e^{it}$, $0 \le t \le \theta$. Note that if $\theta = 2\pi$, we get the circle $\{|z| = \varepsilon\}$, and $\theta = \pi$ corresponds to the semicircle S_ε.

Lemma 4.6.2. *Suppose f has a simple pole at 0. Then*

$$\lim_{\varepsilon\to 0^+}\int_{\gamma_{\varepsilon,\theta}} f = i\theta \operatorname{Res}_0 f.$$

Proof. Let $a = \operatorname{Res}_0 f$. We may write $f(z) = \frac{a}{z} + h(z)$ such that h is holomorphic at 0. We compute

$$\int_{\gamma_{\varepsilon,\theta}} \frac{a}{z}\, dz = \int_0^\theta \frac{a}{\varepsilon e^{it}} i\varepsilon e^{it} dt = \int_0^\theta ia dt = i\theta a.$$

Since h is bounded near 0, and $L(\gamma_{\varepsilon,\theta}) = \varepsilon\theta \to 0$ as $\varepsilon \to 0$, we get $\int_{\gamma_{\varepsilon,\theta}} h \to 0$ as $\varepsilon \to 0^+$. Thus, $\lim_{\varepsilon\to 0^+}\int_{\gamma_\varepsilon} f = i\theta a = i\theta \operatorname{Res}_0 f$. $\square$

Since $\operatorname{Res}_0 \frac{e^{iz}}{z} = 1$, from the lemma we get $\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \frac{e^{iz}}{z}\, dz = i\pi$. Thus,

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{1}{2i}\lim_{\varepsilon\to 0}\int_{S_\varepsilon} \frac{e^{iz}}{z}\, dz = \frac{\pi}{2}.$$

Example. Compute $\int_{-\infty}^\infty \frac{\cos(ax)}{1+x^2}\, dx$ for $a \in \mathbb{R}$.

First we assume that $a \geq 0$. Let $f(z) = \frac{e^{iaz}}{1+z^2}$. Let $R > 1$. Let Γ be the contour $[-R, R] \oplus S_R$. Using Residue formula, we get

$$\int_{-R}^R f(x)dx + \int_{S_R} f = 2\pi i \operatorname{Res}_i f = 2\pi i \frac{e^{iai}}{2i} = e^{-a}\pi.$$

Note that

$$\int_{-R}^R f(x)dx = \int_{-R}^R \frac{\cos(ax)}{1+x^2}\, dx + i\int_{-R}^R \frac{\sin(ax)}{1+x^2} = \int_{-R}^R \frac{\cos(ax)}{1+x^2}\, dx.$$

The second term vanishes because $\frac{\sin(ax)}{1+x^2}$ is odd.

For $z \in S_R$, $\operatorname{Im} z \geq 0$, so $\operatorname{Re}(iaz) \leq 0$. Thus, $|e^{iaz}| = e^{\operatorname{Re}(iaz)} \leq 1$ on S_R. So $\|f\|_{S_R} \leq \frac{1}{R^2-1}$, which implies that

$$\|\int_{S_R} f\| \leq \|f\|_{S_R} L(S_R) \leq \frac{\pi R}{R^2-1} \to 0,$$

as $R \to \infty$. Thus, letting $R \to \infty$ and using the fact that $\frac{\cos(ax)}{1+x^2}$ is even, we conclude that

$$\int_{-\infty}^\infty \frac{\cos(ax)}{1+x^2}\, dx = e^{-a}\pi.$$

Now we consider the case $a < 0$. The above argument does not work because $|e^{iaz}| \geq 1$ on S_R. You may work on a lower semi circle, and repeat the above argument. There is a simple way to do this. Using the fact that cos is even, we get

$$\int_{-\infty}^\infty \frac{\cos(ax)}{1+x^2}\, dx = \int_{-\infty}^\infty \frac{\cos(-ax)}{1+x^2}\, dx = e^{a}\pi,$$

where the second equality holds because $-a > 0$. Thus, we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\, dx = e^{-|a|}\pi, \quad a \in \mathbb{R}.$$

Remark. The Fourier Transformation of f is the function $\widehat{f}$ defined to be

$$\widehat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx, \quad t \in \mathbb{R}.$$

The above computation shows that, for $f(x) = \frac{1}{1+x^2}$, $\widehat{f}(t) = e^{-|t|}\pi$.

Example. Find the Fourier transform of $f(x) = e^{-x^2/2}$.

First, we calculate $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$. Since $e^{-\frac{x^2}{2}} > 0$, applying Fubini's Theorem and using polar coordinate, we get

$$\Big(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\Big)^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \cdot \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta = 2\pi e^{-\frac{r^2}{2}}\Big|_{\infty}^{0} = 2\pi.$$

So we get $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$.

Now we calculate $\widehat{f}(a) = \int_{-\infty}^{\infty} e^{iax} e^{-\frac{x^2}{2}} dx$. Let $f(z) = e^{-\frac{z^2}{2}}$. We have

$$\widehat{f}(a) = \int_{-\infty}^{\infty} e^{-\frac{(x-ia)^2}{2}} e^{-\frac{a^2}{2}} dx = e^{-\frac{a^2}{2}} \lim_{R\to\infty} \int_{[-R-ia, R-ia]} f.$$

We will show that $\lim_{R\to\infty} \int_{[-R-ia,R-ia]} f = \lim_{R\to\infty} \int_{[-R,R]} f = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$. Then we get $\widehat{f}(a) = \sqrt{2\pi} e^{-\frac{a^2}{2}}$. In order to do that, we construct a closed curve:

$$\Gamma = [-R, R] \oplus [R, R-ia] \oplus [R-ia, -R-ia] \oplus [-R-ia, -R].$$

Since f is an entire function,

$$0 = \int_\Gamma f = \int_{[-R,R]} f - \int_{[-R-ia,R-ia]} f + \int_{[R,R-ia]} f - \int_{[-R,-R-ia]} f.$$

For $z \in [R, R-ia]$, $\operatorname{Re} z = R$ and $|\operatorname{Im} z| \le |a|$, which implies that $\operatorname{Re}(z^2) \ge R^2 - a^2$. So $|f(z)| = e^{-\operatorname{Re}(z^2)/2} \le e^{-\frac{R^2-a^2}{2}}$ on $[R, R-ia]$. Then we get

$$\Big|\int_{[R,R-ia]} f\Big| \le L([R, R-ia])\|f\|_{[R,R-ia]} \le |a| e^{-\frac{R^2-a^2}{2}} \to 0, \quad R \to \infty.$$

So $\lim_{R\to\infty} \int_{[R,R-ia]} f = 0$. Similarly, $\lim_{R\to\infty} \int_{[-R,-R-ia]} f = 0$. So we get

$$\lim_{R\to\infty} \int_{[-R-ia,R-ia]} f = \lim_{R\to\infty} \int_{[-R,R]} f = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

as desired, and then get $\widehat{f}(a) = \sqrt{2\pi} e^{-\frac{a^2}{2}}$.

Homework Chapter VI §2: 2, 8(a), 9

Trigonometric Integrals We wish to evaluate an integral of the form

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta) d\theta,$$

where Q is a rational function of two variables: $Q = Q(x, y)$. Since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

we find that

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta) d\theta = \int_{|z|=1} \frac{1}{iz} Q\Big(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\Big) dz.$$

We may then use Residue formula to calculate the integral on the right.

Example. Compute $I = \int_0^{2\pi} \frac{1}{a+\sin\theta}\, d\theta$, where $a > 1$.

We have

$$\int_0^{2\pi} \frac{1}{a + \sin\theta}\, d\theta = \int_{|z|=1} \frac{1}{iz} \cdot \frac{1}{a + (z - z^{-1})/(2i)}\, dz$$
$$= \int_{|z=1} \frac{2}{z^2 + i2az - 1}\, dz = 2\pi i \operatorname{Res}_{z_0} \frac{2}{z^2 + i2az - 1},$$

where z_0 is a root of $z^2 + i2az - 1 = (z + ia)^2 - 1 + a^2$ inside $\{|z| < 1\}$, which is $i(\sqrt{a^2 - 1} - a)$, and

$$\operatorname{Res}_{z_0} \frac{2}{z^2 + i2az + 1} = \frac{2}{2z + i2a|_{z=z_0}} = \frac{1}{i\sqrt{a^2 - 1}}.$$

Thus, $I = \frac{2\pi}{\sqrt{a^2-1}}$.

Remark. If we integrate from $-\pi$ to π, then we get the same value. If $Q(\cos\theta, \sin\theta)$ is even, then $\int_0^\pi = \frac{1}{2}\int_{-\pi}^\pi = \frac{1}{2}\int_0^{2\pi}$.

We end this section with an integral, which uses a branch of logarithm.

Example. Compute $\int_0^\infty \frac{1}{1+x^a}\, dx$ for $a > 1$.

Note that the improper integral converge by comparison principle: $\frac{1}{1+x^a} < \frac{1}{x^a}$. If $a \le 1$, the integral is ∞. Let $f(z) = \frac{1}{1+z^a} = \frac{1}{1+e^{aL(z)}}$, where L is a branch of logarithm. We need to specify the branch to make f well defined. We first define a contour. For $r > 0$, let A_r denote the curve $\gamma(t) = re^{it}$, $0 \le t \le 2\pi/a$. Fix $R > 1 > \varepsilon > 0$. Let

$$\Gamma = [\varepsilon, R] \oplus A_R \oplus [Re^{i2\pi/a}, \varepsilon e^{i2\pi/a}] \oplus A_\varepsilon^-.$$

Then Γ is a Jordan curve. We want to choose a simply connected domain $G \subset \mathbb{C} \setminus \{0\}$ such that $\Gamma \cup \mathrm{Int}(\Gamma) \subset G$. Then $\log z$ has a branch, which is holomorphic on G. A nice choice is $G = \mathbb{C} \setminus \{re^{i\pi(1/a+1)} : r \ge 0\}$. We now choose the branch L of log in G such that $L(x) = \log(x)$ if $x \in \mathbb{R}$ and $x > 0$. Then we have $\mathrm{Im}\, L(z) \in [0, 2\pi/a]$ on Γ and its interior.

To find the singularity of f, we solve $0 = 1 + z^a = 1 + e^{aL(z)}$, which gives $aL(z) = i\pi + i2n\pi$, $n \in \mathbb{Z}$. Since $\mathrm{Im}\, L(z) \in [0, 2\pi/a]$ on and inside Γ, the only singularity z_0 inside Γ satisfies $L(z_0) = i\frac{\pi}{a}$, and so $z_0 = e^{i\frac{\pi}{a}}$. Now we have

$$\mathrm{Res}_{z_0} f = \frac{1}{az_0^a/z_0} = -\frac{z_0}{a}.$$

We may reparameterize the curve $[\varepsilon, R]$ using $\gamma(x) = x$, $\varepsilon \le x \le R$. Then we see that $\int_{[\varepsilon,R]} f = \int_\varepsilon^R \frac{1}{1+x^a} dx$. We may reparameterize $[\varepsilon e^{i2\pi/a}, Re^{i2\pi/a}]$ by $\gamma(x) = xe^{i2\pi/a}$, $\varepsilon \le x \le R$. Then we see that

$$\int_{[\varepsilon e^{i2\pi/a}, Re^{i2\pi/a}]} f = \int_\varepsilon^R \frac{e^{i2\pi/a}}{1 + (xe^{i2\pi/a})^a}\, dx = e^{i2\pi/a} \int_\varepsilon^R \frac{1}{1+x^a} dx.$$

Thus, from Residue formula,

$$\int_\varepsilon^R \frac{dx}{1+x^a} + \int_{A_R} f - e^{i2\pi/a} \int_\varepsilon^R \frac{dx}{1+x^a} - \int_{A_\varepsilon} f = 2\pi i \frac{-e^{i\pi/a}}{a}.$$

We will show that $\int_{A_R} f \to 0$ as $R \to \infty$ and $\int_{A_\varepsilon} f \to 0$ as $\varepsilon \to 0$. Then we get

$$\int_0^\infty \frac{dx}{1+x^a} = \frac{2\pi i}{a} \frac{e^{i\pi/a}}{e^{i2\pi/a} - 1} = \frac{\pi}{a} \frac{2i}{e^{i\pi/a} - e^{-i\pi/a}} = \frac{\pi}{a \sin(\pi/a)}.$$

Note that $|z^a| = |e^{aL(z)}| = e^{\mathrm{Re}\, aL(z)} = e^{a \ln |z|} = |z|^a$. For $z \in A_R$, $|1 + z^a| \ge |z^a| - 1 = |z|^a - 1 = R^a - 1 > 0$. So $\|f\|_{A_R} \le \frac{1}{R^a - 1}$. Thus, as $R \to \infty$,

$$\Big| \int_{A_R} f \Big| \le \|f\|_{A_R} L(A_R) \le \frac{2\pi R}{R^a - 1} \to 0,$$

where we used $a > 1$. For $z \in A_\varepsilon$, $|1 + z^a| \ge 1 - |z|^a = 1 - \varepsilon^a > 0$. So $\|f\|_{A_\varepsilon} \le \frac{1}{1-\varepsilon^a}$. Thus, as $\varepsilon \to 0$,

$$\Big| \int_{A_\varepsilon} f \Big| \le \|f\|_{A_\varepsilon} L(A, \varepsilon) \le \frac{2\pi\varepsilon}{1 - \varepsilon^a} \to 0.$$

Homework Chapter VI §2: 11, 14(a), 19, 24.

4.7 Multiplication and Division of Power Series

Suppose f and g are holomorphic at z_0, or have z_0 as a removable singularity or pole. Suppose the power series expansion (including possible negative power terms) of f and g centered at z_0 are known. We will study the method to calculate the coefficients of the power series of fg and f/g. This calculation will also help us to find residues of some functions.

For simplicity, assume $z_0 = 0$. Assume that neither f or g is constant 0 near 0. Let m_1 and m_2 be the order of f and g, respectively, at 0. Then we may write $f(z) = \sum_{n=m_1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=m_2}^{\infty} b_n z^n$ near 0 such that $a_{m_1}, b_{m_2} \neq 0$. Then the product fg has order $m_1 + m_2$ at 0, and it has a power series $\sum_{n=m_1+m_2}^{\infty} c_n z^n$ near 0, where the coefficients c_n can be expressed in terms of a_n and b_n:

$$c_n = \sum_{(j,k):j+k=n} a_j b_k, \quad n \geq m_c.$$

Note that every sum is essentially a finite sum. For example, if $m_1 = m_2 = 0$, then the first several equalities are $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b_0$, $c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$, $c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0$.

The division f/g has order $m_1 - m_2$ at 0, and it has a power series, say $\sum_{n=m_1-m_2}^{\infty} d_n z^n$ near 0. To find the coefficients d_n, we may use that $f = g \cdot (f/g)$ and the equalities in the last paragraph. We can calculate d_n by solving linear equations. To make it simple, suppose $m_1 = m_2 = 0$. Then the first several equations are: $b_0 d_0 = a_0$, $b_0 d_1 + b_1 d_0 = a_1$, $b_0 d_2 + b_1 d_1 + b_2 d_0 = a_2$, $b_0 d_3 + b_1 d_2 + b_2 d_1 + b_3 d_0$. Solving the first equation, we get d_0; plugging d_0 into the second equation, we get d_1; plugging d_0 and d_1 into the third equation, we get d_2; and so on. We may also do the computation using a division algorithm, which is similar to those used in elementary arithmetic.

Example Let $f(z) = \frac{1}{1-z}$ and $g(z) = -\log(1-z)$. We wish to calculate the power series for f/g centered at 0. Note that $f(z) = \sum_{n=0}^{\infty} z^n$ and $g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ near 0. The leading nonzero term in the power series of f/g is z^{-1}. Suppose the power series is $z^{-1} + c_0 + c_1 z + c_2 z^2 + \cdots$. Then we have $(z^{-1} + c_0 + c_1 z + c_2 z^2 + \cdots)(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots) = 1 + z + z^2 + z^3 + \cdots$. Then we find $c_0 + \frac{1}{2} = 1$, $c_1 + \frac{c_0}{2} + \frac{1}{3} = 1$, $c_2 + \frac{c_1}{2} + \frac{c_0}{3} + \frac{1}{4} = 1$, and so on. Solving these equations, we get $c_0 = \frac{1}{2}$, $c_1 = \frac{5}{12}$, $c_2 = \frac{3}{8}$. We may use this result to compute residues. For example, since $\operatorname{Res}_0 \frac{f(z)/g(z)}{z^3}$ equals to the coefficient c_2, which is $\frac{3}{8}$.

Homework

1. Find the coefficients of the Laurent series of $\cot z$ centered at 0 up to z^3.

2. (20 points) (i) Compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by integrating $f(z) = \frac{\cot z}{z^2}$ along the boundary of a square with vertices $\{C_N + iC_N, -C_N + iC_N, -C_N - iC_N, C_N - iC_N\}$, where $C_N = (N + \frac{1}{2})\pi$ and $N \in \mathbb{N}$, and letting $N \to \infty$. (ii) Sketch the computation of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ and give the answer.

   Note: You need to prove that the integral of f along that boundary tends to 0 as $N \to \infty$.

Chapter 5

Conformal Mappings

In this chapter we study analytic isomorphisms. An analytic isomorphism is also called a conformal map. We say that f is an analytic isomorphism of U with V if f is an analytic isomorphism defined on U, and $f(U) = V$. Let $\mathrm{Iso}(U, V)$ denote the set of all analytic isomorphisms of U with V. If $f \in \mathrm{Iso}(U, V)$, then $f^{-1} \in \mathrm{Iso}(V, U)$. If $f \in \mathrm{Iso}(U, V)$ and $g \in \mathrm{Iso}(V, W)$, then $g \circ f \in \mathrm{Iso}(U, W)$. Recall that an analytic isomorphism of U with itself is an analytic automorphism of U. Let $\mathrm{Aut}(U) = \mathrm{Iso}(U, U)$ denote the set of all analytic automorphisms of U. Then $\mathrm{Aut}(U)$ has a clear group structure.

The name of conformal maps come from the following observation. Let U be an open set in $\mathbb{C}$ and let $\gamma : [a, b] \to U$ be a C^1 curve in U. Let $f : U \to \mathbb{C}$ be holomorphic, and $\beta = f \circ \gamma$, i.e., $\beta(t) = f(\gamma(t))$. Then we have $\beta'(t) = f'(\gamma(t))\gamma'(t)$. We interpret $\gamma'(t)$ as a vector in the direction of a tangent vector at the point $\gamma(t)$. If $\gamma'(t) \neq 0$, it defines the direction of the curve at the point.

Suppose that two differentiable curves γ and η intersect at $z_0 = \gamma(t_0) = \eta(t_1)$. Suppose $\gamma'(t_0)$ and $\eta'(t_1)$ are not 0. The angle θ between the two tangent vectors $\gamma'(t_0)$ and $\eta'(t_1)$ is defined to be the angle between γ and η at z_0. We may write $\theta = \arg \gamma'(t_0) - \arg \eta'(t_1)$. If $f'(z_0) \neq 0$, then

$$\arg \frac{d}{dt} f(\gamma(t_0)) = \arg f'(z_0) + \arg \gamma'(t_0), \quad \arg \frac{d}{dt} f(\eta(t_1)) = \arg f'(z_0) + \arg \eta'(t_1).$$

So we see that the angle between the curves $f \circ \gamma$ and $f \circ \eta$ at $f(z_0)$ is the same as the angle between γ and η at z_0.

Suppose f is a conformal map defined on U. Since f' never vanishes, f preserves the angle between any two curves in U. So in some sense, the shape of a subset $S \subset U$ is similar as its image $f(U)$ under f. The inverse map f^{-1} has a similar property.

5.1 Schwarz Lemma

Let $\mathbb{D} = D(0, 1)$ denote the open unit disc. For $c \in \mathbb{C}$, define $M_c(z) = cz$. It is clear that, if $|c| = 1$, then $M_c \in \mathrm{Aut}(\mathbb{D})$.

Theorem 5.1.1. [Schwarz Lemma] *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function such that $f(0) = 0$. Then*

(i) $|f(z)| \le |z|$ for all $z \in \mathbb{D}$, and $|f'(0)| \le 1$.

(ii) If for some $z_0 \in \mathbb{D} \setminus \{0\}$, $|f(z_0)| = |z_0|$, or $|f'(0)| = 1$, then $f = M_c$ for some $|c| = 1$. In particular, if $f(z_0) = z_0$ or $f'(0) = 1$, then $c = 1$, i.e., $f = \mathrm{id}$.

Proof. Since 0 is a zero of f, it is a removable singularity of $f(z)/z$. Let $h(z) = f(z)/z$ for $z \in \mathbb{D} \setminus \{0\}$ and $h(0) = \lim_{z\to 0} f(z)/z = f'(0)$. Then h is also analytic in $\mathbb{D}$. For any $r \in (0, 1)$, $|h(z)| = |f(z)|/|z| < 1/r$ on $\{|z| = r\}$. From Maximum Modulus Principle, $|h(z)| \le 1/r$ on $\{|z| \le r\}$. Fix any $z \in \mathbb{D}$. If $|h(z)| > 1$, then we may find $r \in (0, 1)$ such that $r > \max\{|z|, 1/|h(z)|\}$. However, from $|z| < r$, we should get $|h(z)| \le 1/r$, which contradicts that $r > 1/|h(z)|$. Thus, $|h(z)| \le 1$ for $z \in \mathbb{D}$. Since $f(z) = zh(z)$, we get $|f(z)| \le |z|$ for $z \in \mathbb{D}$. Since $f'(0) = h(0)$, we get $|f'(0)| \le 1$. If the condition in (ii) is satisfied, then $|h|$ attains its maximum at an interior point, which implies that h is constant c in $\mathbb{D}$, and $|c| = 1$. Thus, $f(z) = cz$. The rest of the statement is obvious. □

Remark. If $f \in \mathrm{Aut}(\mathbb{D})$ and $f(0) = 0$, then by applying Schwarz lemma twice to f and f^{-1}, we can conclude that $f = M_c$ for some $|c| = 1$.

Homework.

1. Let $f \in \mathrm{Iso}(\mathbb{D}, U)$. Show that if $D(f(0), R) \subset U$, then $R \le |f'(0)|$. Hint: Consider the restriction of f^{-1} to $D(f(0), R)$.

5.2 Analytic Automorphisms of the Disc

Let $\alpha \in \mathbb{D}$. Let

$$g(z) = g_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

Since g has only one pole, which is $1/\bar{\alpha}$ that lies outside $\{|z| \le 1\}$, we see that g is analytic on $\{|z| \le 1\}$. If $|z| = 1$, then

$$|g(z)| = \frac{|\alpha - z|}{|1 - \bar{\alpha} z|} = \frac{1}{|z|} \frac{|\alpha - z|}{|1/z - \bar{\alpha}|} = \frac{|\alpha - z|}{|\bar{z} - \bar{\alpha}|} = 1.$$

From Maximum Modulus Principle, $|g(z)| \le 1$ for $z \in \mathbb{D}$. In fact, we have $|g(z)| < 1$ for $z \in \mathbb{D}$ because if $|g(z_0)| = 1$ for some $z_0 \in \mathbb{D}$, then g has to be a constant, which is not true. Thus, g maps $\mathbb{D}$ into $\mathbb{D}$. If $w = g_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$, then $w - w\bar{\alpha} z = \alpha - z$, and $z = \frac{\alpha - w}{1 - \bar{\alpha} w} = g_\alpha(w)$. This shows that $g_\alpha^{-1} = g_\alpha$. Thus, $g \in \mathrm{Aut}(\mathbb{D})$.

Theorem 5.2.1. *Every $f \in \mathrm{Aut}(\mathbb{D})$ can be expressed by $f = M_c \circ g_\alpha$ for some $c \in \{|z| = 1\}$ and $\alpha \in \mathbb{D}$.*

Proof. First, since $M_c, g_\alpha \in \mathrm{Aut}(\mathbb{D})$, we have $M_c \circ g_\alpha \in \mathrm{Aut}(\mathbb{D})$. Now suppose $f \in \mathrm{Aut}(\mathbb{D})$. Let $\alpha = f^{-1}(0) \in \mathbb{D}$ and $h = f \circ g_\alpha$. Then $h \in \mathrm{Aut}(\mathbb{D})$, and $h(0) = f(g_\alpha(0)) = f(\alpha) = 0$. From the remark after Schwarz lemma, we see that $h = M_c$ for some $|c| = 1$. Thus, $f = M_c \circ g_\alpha$. $\square$

Remark. Combining Schwarz Lemma with the map g_α, we can obtain some inequalities of analytic maps $f : \mathbb{D} \to \mathbb{D}$. For example, if $z \in \mathbb{D}$ and $w = f(z) \in \mathbb{D}$, then the composition $h := g_w \circ f \circ g_z$ satisfies the condition of Schwarz lemma. We get inequalities for h by applying Schwarz lemma. Then we can obtain inequalities for f.

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5.3 The Upper Half Plane

We use $\mathbb{H}$ to denote the open upper half plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$.

Theorem 5.3.1. *Let* $f(z) = \frac{z-i}{z+i}$. *Then* $f \in \mathrm{Iso}(\mathbb{H}, \mathbb{D})$.

Proof. Note that f is a rational function with only one pole at $-i$, which lies outside $\mathbb{H}$. Let $z \in \mathbb{H}$. Write $z = x + iy$ with $y > 0$. Then

$$|z-i|^2 = x^2 + (y-1)^2 = x^2 + y^2 + 1 - 2y < x^2 + y^2 + 1 + 2y = x^2 + (y+1)^2 = |z+i|^2.$$

Thus, $|f(z)| = \frac{|z-i|}{|z+i|} < 1$. So f maps $\mathbb{H}$ into $\mathbb{D}$. Suppose $w = f(z)$. We may solve z in terms of w: $z = -i\frac{w+1}{w-1}$. So f is injective. In addition, we claim that $z \in \mathbb{H}$ if $w \in \mathbb{D}$, which finishes the proof.

Write $w = re^{i\theta} = r\cos\theta + ir\sin\theta$ for some $r \in [0, 1)$ and $\theta \in \mathbb{R}$. Then

$$\begin{aligned} z = -i\frac{r\cos\theta + 1 + ir\sin\theta}{r\cos\theta - 1 + ir\sin\theta} &= \frac{-i(r\cos\theta + 1 + ir\sin\theta)(r\cos\theta - 1 - ir\sin\theta)}{(r\cos\theta - 1 + ir\sin\theta)(r\cos\theta - 1 - ir\sin\theta)} \\ &= \frac{-i((r\cos\theta)^2 - (1 + ir\sin\theta)^2)}{(r\cos\theta - 1)^2 + (r\sin\theta)^2} = \frac{-2r\sin\theta + i(1-r^2)}{(r\cos\theta - 1)^2 + (r\sin\theta)^2}. \end{aligned}$$

Thus, $\mathrm{Im}\, z = \frac{1-r^2}{(r\cos\theta-1)^2 + (r\sin\theta)^2} > 0$, which implies that $z \in \mathbb{H}$. $\square$

There are some obvious analytic automorphisms of $\mathbb{H}$. If $a \in \mathbb{R}$, then $f(z) = z + a \in \mathrm{Aut}(\mathbb{H})$. If $c > 0$, then $f(z) = cz \in \mathrm{Aut}(\mathbb{H})$. Thus, $f(z) = cz + a \in \mathrm{Aut}(\mathbb{H})$ for $c > 0$ and $a \in \mathbb{R}$.

Let $z_0 = x_0 + iy_0 \in \mathbb{H}$ and $h_{z_0}(z) = \frac{z - z_0}{z - \overline{z_0}}$. Note that $h_i(z) = \frac{z-i}{z+i} \in \mathrm{Iso}(\mathbb{H}, \mathbb{D})$. We may write

$$h_{z_0}(z) = \frac{z - x_0 - iy_0}{z_0 - x_0 + iy_0} = \frac{\frac{z-x_0}{y_0} - i}{\frac{z-x_0}{y_0} + i} = h_i((z - x_0)/y_0).$$

Since $z \mapsto \frac{z-x_0}{y_0} \in \mathrm{Aut}(\mathbb{H})$ and $h_i \in \mathrm{Iso}(\mathbb{H}, \mathbb{D})$, we see that $h_{z_0} \in \mathrm{Iso}(\mathbb{H}, \mathbb{D})$.

Remarks.

1. Every $f \in \text{Aut}(\mathbb{H})$ can be expressed by $f = h_{z_2}^{-1} \circ M_c \circ h_{z_1}$ for some $z_1, z_2 \in \mathbb{H}$ and $c \in \mathbb{C}$ with $|c| = 1$. In fact, choose any $z_1 \in \mathbb{H}$ and let $z_2 = f(z_1) \in \mathbb{H}$. Then $g := h_{z_2} \circ f \circ h_{z_1}^{-1} \in \text{Aut}(\mathbb{D})$ and fixes 0. Thus, $g = M_c$ for some $c \in \mathbb{C}$ with $|c| = 1$.

2. Combining Schwarz Lemma with the map h_{z_0}, we can obtain some inequalities of analytic maps $f : \mathbb{H} \to \mathbb{H}$.

Homework.

1. Suppose $f : \mathbb{H} \to \mathbb{H}$ is analytic. Prove that

   (i)
$$\Big|\frac{f(z) - f(w)}{f(z) - \overline{f(w)}}\Big| \le \Big|\frac{z - w}{z - \overline{w}}\Big|, \quad \forall z, w \in \mathbb{H}.$$

   (ii) If $f \in \text{Aut}(\mathbb{H})$, then the equality in the above formula holds for any $z, w \in \mathbb{H}$.

   (iii) If the equality in the above formula holds for any one pair $z_0 \neq w_0 \in \mathbb{H}$, then $f \in \text{Aut}(\mathbb{H})$.

5.4 Riemann Sphere

We now add an extra element ∞ to $\mathbb{C}$, and call $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the extended complex plane. Unlike the extended real line $[-\infty, \infty]$, we here only need one extra element. We define a disc centered at ∞ with radius $r > 0$ to be $D(\infty, r) := \{\infty\} \cup \{z \in \mathbb{C} : |z| > 1/r\}$. A subset U of $\widehat{\mathbb{C}}$ is called open if for every $z \in U$, there is $r > 0$ such that $D(z, r) \subset U$. This means that (i) $U \cap \mathbb{C}$ is an open set in $\mathbb{C}$; and (ii) if $\infty \in U$, then for some $R > 0$, $\{|z| > R\} \subset U$. Moreover, that $z \to \infty$ means that $|z| \to \infty$. $\widehat{\mathbb{C}}$ is also called the Riemann sphere. It is called a sphere because $\widehat{\mathbb{C}}$ is homeomorphic to the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. A typical homeomorphism $h : S^2 \to \mathbb{C}$ is given by the stereographic projection: $h(x, y, z) = \frac{x+iy}{1-z}$ if $z < 1$ and $h(0, 0, 1) = \infty$. The Riemann sphere may be viewed as a one-dimensional complex manifold.

The Riemann sphere $\widehat{\mathbb{C}}$ gives a new description of simply connected domains: for a domain $D \subset \mathbb{C}$, D is simply connected if and only if $\widehat{\mathbb{C}} \setminus D$ is connected. Here a set $K \subset \widehat{\mathbb{C}}$ is called connected if there are no two open sets U_1 and U_2 in $\widehat{\mathbb{C}}$ such that $U_1 \cap U_2 = \emptyset$, $K \subset U_1 \cup U_2$, and $K \cap U_j \neq \emptyset$ for $j = 1, 2$.

The $\widehat{\mathbb{C}}$ in the above statement can not be replaced with $\mathbb{C}$. For example, $D = \mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$ is a simply connected domain, and $\widehat{\mathbb{C}} \setminus D = (-\infty, 0] \cup [1, \infty) \cup \{\infty\}$ is connected, but $\mathbb{C} \setminus D = (-\infty, 0] \cup [1, \infty)$ is not connected. Another example is $D = \mathbb{C} \setminus [0, 1]$, which is not simply connected. Note that $\widehat{\mathbb{C}} \setminus D = [0, 1] \cup \{\infty\}$ is not connected, but $\mathbb{C} \setminus D = [0, 1]$ is connected.

This statement also motivates one to define multiply connected domains. We say that a domain $D \subset \mathbb{C}$ is n-connected, if $\widehat{\mathbb{C}} \setminus D$ has n connected components. For example, an annulus is 2-connected.

Let $U \subset \mathbb{C}$ be an open set. A map $f : U \to \widehat{\mathbb{C}}$ is called an extended analytic function if for every $z_0 \in U$, either $f(z_0) \in \mathbb{C}$ and f is analytic at z_0 in the usual sense, or $f(z_0) = \infty$ and $\frac{1}{f}$ is analytic at z_0 with the convention that $\frac{1}{\infty} = 0$. This means that, if $f(z_0) \neq \infty$, then there exists $r > 0$ such that $f(z) \neq \infty$ on $D(z_0, r)$; if $f(z_0) = \infty$, then there is $r > 0$ such that either $f \equiv \infty$ on $D(z_0, r)$, or $f \neq \infty$ on $D(z_0, r) \setminus \{z_0\}$. In the last case, z_0 is a pole of f. Thus, if U is a domain, then either f is constant ∞, or the set $f^{-1}(\infty)$ has no accumulation points in U. In the latter case, f is meromorphic in U with poles at $f^{-1}(\infty)$. On the other hand, if f is meromorphic in U, then by defining the value of f at each pole to be ∞, we get an extended analytic function in U.

Let $U \subset \widehat{\mathbb{C}}$ be an open set. A map $f : U \to \widehat{\mathbb{C}}$ is called an extended analytic function if (i) f restricted to $U \cap \mathbb{C}$ is an extended analytic function defined above; and (ii) if $\infty \in U$, then the function g defined by $g(z) = f(1/z)$ is extended analytic at 0. Here we use the convention that $\frac{1}{0} = \infty$.

Example. Let $R = P/Q$ be a rational function. First, it is a meromorphic function on $\mathbb{C}$. Second, note that $g(z) := R(1/z)$ is also a rational function if we define $g(0)$ in a suitable way. Then if we define $R(\infty) = g(0)$, we get that R is extended analytic at ∞. So every rational function is an extended analytic function on $\widehat{\mathbb{C}}$.

In the same spirit, we may talk about the singularity at ∞. We say that ∞ is a singularity of f if f is analytic in $\{|z| > R\}$ for some $R > 0$. This is the case, for example, if f is an entire function. If we define $g(z) = f(1/z)$, then 0 is a singularity of g. The type of the singularity ∞ of f is then defined as the type of 0 of g. Suppose that the Laurent series expansion of f in $\{R < |z| < \infty\}$ is $\sum_{n=-\infty}^{\infty} a_n z^n$. Then ∞ is removable if $a_n = 0$ for all $n > 0$; is essential if there exist infinitely many $n \in \mathbb{N}$ such that $a_n \neq 0$; and is a pole of order $n \in \mathbb{N}$ if n is the biggest number such that $a_n \neq 0$. If ∞ is an essential singularity, then for any $T > R$, the image $f(\{|z| > T\})$ is dense in $\mathbb{C}$.

In particular, if f is an entire function, then ∞ is a singularity of f. Since the a_n is zero if $n < 0$, we find that, if ∞ is removable, then f is constant; if ∞ is a pole, then f is a nonconstant polynomial.

Lemma 5.4.1. *Every analytic isomorphism f of $\mathbb{C}$ has the form $f(z) = az+b$ for some $a, b \in \mathbb{C}$ with $a \neq 0$.*

From this lemma, we see that $\mathrm{Iso}(\mathbb{C}, U) = \emptyset$ if $U \neq \mathbb{C}$, and $\mathrm{Iso}(\mathbb{C}, \mathbb{C})$ is composed of polynomials of degree 1.

Proof. Now f is an entire function. Consider the type of singularity ∞ of f. Since f is not constant, ∞ is not removable. If ∞ is an essential singularity, then $f(\{|z| > 1\})$ is dense in $\mathbb{C}$. However, since f is injective, $f(\{|z| > 1\})$ is disjoint from $f(\{|z| < 1\})$. From the open mapping theorem, $f(\{|z| < 1\})$ is an open set. Thus, the closure of $f(\{|z| > 1\})$ is contained in $\mathbb{C} \setminus f(\{|z| < 1\})$, which contradicts that $f(\{|z| > 1\})$ is dense in $\mathbb{C}$. So ∞ is a pole of f, which implies that f is a polynomial, so f' is also a polynomial. Since f is injective, f' has no zero in

$\mathbb{C}$. From the Fundamental Theorem of Algebra, f' must be a nonzero constant, say a, which implies that $f(z)$ has the form $az + b$. $\square$

It is not difficult to check that, if f and g are extended analytic functions defined on open sets $U, V \subset \widehat{\mathbb{C}}$, respectively, and $f(U) \subset V$, then $g \circ f$ is an extended analytic function defined on U. If $V = f(U)$ and f is injective on U, then we say that f is an extended analytic isomorphism of U with V. If $V = U$, we say that f is an extended analytic automorphism of U. We still use the symbols $\mathrm{Iso}(U, V)$ and $\mathrm{Aut}(U)$ to denote the set of extended analytic isomorphisms and extended analytic automorphisms.

5.5 Mobius Transformation

Let $\mathrm{GL}_2(\mathbb{C})$ denote the set of 2×2 complex matrices with nonzero determinant. For every $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$, we define a rational function.

$$f_M(z) = \frac{az+b}{cz+d}.$$

Note that if $M = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $f(z) = z$ is the identity function. Also note that $f_{rM} = f_M$ for any $r \in \mathbb{C} \setminus \{0\}$. Since f is a rational function, we may view it as an extended analytic function defined on $\widehat{\mathbb{C}}$. If $c = 0$, then $ad \neq 0$ and $f_M = \frac{a}{d}z + \frac{b}{d}$ is a polynomial of degree 1, and $f_M(\infty) = \infty$). If $c \neq 0$, then $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$.

Definition 5.5.1. *Every f_M is called a Möbius transformation or a fractional linear transformation.*

Examples.

1. For $a, b \in \mathbb{C}$ with $a \neq 0$, let $M = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$. Then $f_M(z) = az + b$ is a polynomial of degree 1, and $f_M(\infty) = \infty$.

2. For $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $f_M(z) = \frac{1}{z}$.

3. Let $\alpha \in \mathbb{D}$ and $M = \begin{bmatrix} -1 & \alpha \\ -\overline{\alpha} & 1 \end{bmatrix}$. Then $f_M = g_\alpha \in \mathrm{Aut}(\mathbb{D})$.

4. For $z_0 \in \mathbb{H}$, the function $h_{z_0}(z) = \frac{z - z_0}{z - \overline{z_0}}$ can be expressed by f_M, where $M = \begin{bmatrix} 1 & -z_0 \\ 1 & -\overline{z_0} \end{bmatrix}$.

Consider $M_1, M_2 \in \mathrm{GL}_2(\mathbb{C})$:

$$M_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad M = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

If $z \in \mathbb{C}$, $c_2 z + d_2 \neq 0$ and $(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2) \neq 0$, then

$$f_{M_1} \circ f_{M_2}(z) = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} = \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)}$$

$$= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)} = f_{M_1 \cdot M_2}(z),$$

where $M_1 \cdot M_2$ is the matrix product of M_1 and M_2, which belongs to $\mathrm{GL}_2(\mathbb{C})$ because $\det(M_1 \cdot M_2) = \det(M_1)\det(M_2) \neq 0$. So we have $f_{M_1 \cdot M_2} = f_{M_1} \circ f_{M_2}$ on $\widehat{\mathbb{C}}$ with at most three possible exceptions (including ∞). In fact, there are no exceptions because both $f_{M_1 \cdot M_2}$ and $f_{M_1} \circ f_{M_2}$ are continuous on $\widehat{\mathbb{C}}$. Thus, $f_{M_1 \cdot M_2} = f_{M_1} \circ f_{M_2}$ holds everywhere on $\widehat{\mathbb{C}}$.

For every $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$, the inverse matrix $M^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ also belongs to $\mathrm{GL}_2(\mathbb{C})$. Thus,

$$f_M \circ f_{M^{-1}} = f_{M^{-1}} \circ f_M = f_{I_2} = \mathrm{id},$$

which implies that $f_M \in \mathrm{Aut}(\widehat{\mathbb{C}})$ and

$$f_M^{-1} = f_{M^{-1}} = f_{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}.$$

Theorem 5.5.1. *Every f in $\mathrm{Aut}(\mathbb{D})$ or $\mathrm{Aut}(\mathbb{H})$ is a Möbius transformation.*

Proof. Let $f \in \mathrm{Aut}(\mathbb{D})$. Then f can be expressed as $M_c \circ g_\alpha$, where $|c| = 1$ and $\alpha \in \mathbb{D}$. Since both M_c and g_α are Möbius transformations, so is f. Now suppose $f \in \mathrm{Aut}(\mathbb{H})$. Recall that $h_i(z) = \frac{z-i}{z+i} \in \mathrm{Iso}(\mathbb{H}, \mathbb{D})$. Let $g = h_i \circ f \circ h_i^{-1}$. Then $g \in \mathrm{Aut}(\mathbb{D})$ and $f = h_i^{-1} \circ g \circ h_i$. Since g, h_i, and h_i^{-1} are all Möbius transformations, so is f. $\square$

Theorem 5.5.2. *Every $f \in \mathrm{Aut}(\widehat{\mathbb{C}})$ is also a Möbius transformation.*

Proof. Let $f \in \mathrm{Aut}(\widehat{\mathbb{C}})$. If $f(\infty) = \infty$, then $f|_{\mathbb{C}} \in \mathrm{Aut}(\mathbb{C})$. We have a lemma, which says that every element in $\mathrm{Aut}(\mathbb{C})$ is a polynomial of degree 1, and so is a Möbius transformation. If $f(\infty) = z_0 \in \mathbb{C}$, let $h(z) = \frac{1}{z - z_0}$ and $g = h \circ f$. Then h is a Möbius transformation with $h(z_0) = \infty$. So $g \in \mathrm{Aut}(\widehat{\mathbb{C}})$ with $g(\infty) = \infty$. From the above, we know that g is a Möbius transformation. Thus, $f = h^{-1} \circ g$ is a Möbius transformation. $\square$

We now define some simple Möbius transformations.

1. $M_a(z) = az$, called multiplication by $a \in \mathbb{C} \setminus \{0\}$;

2. $T_b(z) = z + b$, called translation by $b \in \mathbb{C}$;

3. $J(z) = \frac{1}{z}$, called inversion.

Lemma 5.5.1. *Every Möbius transformation f can be expressed as a composition of simple Möbius transformations.*

Proof. Suppose $f(z) = \frac{az+b}{cz+d}$. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$. So $f = T_{\frac{b}{d}} \circ M_{\frac{a}{d}}$. Now suppose $c \neq 0$. Then $f(z) - \frac{a}{c} = \frac{b'}{cz+d}$, where $b' = b - \frac{ad}{c}$. So $\frac{1}{f(z)-a/c} = \frac{c}{b'}z + \frac{d}{b'}$. So $f = T_{\frac{a}{c}} \circ J \circ T_{\frac{d}{b'}} \circ M_{\frac{c}{b'}}$. □

Definition 5.5.2. *A generalized circle on $\widehat{\mathbb{C}}$ is either a circle in $\mathbb{C}$ or the union of $\{\infty\}$ with a straight line in $\mathbb{C}$. A generalized disc on $\widehat{\mathbb{C}}$ is either a disc in $\mathbb{C}$, or the exterior of a circle together with ∞, or a half plane.*

Remarks.

1. A straight line can be viewed as a circle with radius ∞. Every generalized circle C divides $\widehat{\mathbb{C}}$ into two generalized discs.

2. The stereographic projection generates a one-to-one correspondence between circles on the sphere with the generalized circles on $\widehat{\mathbb{C}}$. Those circles passing the north pole correspond to the straight lines in $\mathbb{C}$.

Theorem 5.5.3. *A Möbius transformation maps generalized circles to generalized circles.*

Proof. From the above lemma, it suffices to show that J, T_b, and M_a map generalized circles to generalized circles. This is obviously true for the translations T_b and the multiplications M_a (which are rotations followed by dilations). Now we consider the map $J(z) = \frac{1}{z}$.

The equation of a circle or a straight line in the (x, y)-plane has the form

$$A(x^2 + y^2) + Bx + Cy + D = 0 \tag{5.1}$$

with $A, B, C, D \in \mathbb{R}$ such that not all A, B, C are equal to 0. In fact, if $A = 0$, we get the equation of a straight line; if $A \neq 0$, we get the equation of a circle. We now consider the equation of the image of this set under $J(z) = \frac{1}{z}$.

Suppose $u + iv = J(x + iy)$. Then $x + iy = J(u + iv)$. We then get $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$. Thus, u and v satisfy the equation

$$A\Big[\Big(\frac{u}{u^2+v^2}\Big)^2 + \Big(\frac{-v}{u^2+v^2}\Big)^2\Big] + B\frac{u}{u^2+v^2} + C\frac{-v}{u^2+v^2} + D = 0.$$

Multiplying the formula by $u^2 + v^2$, we get

$$A + Bu - Cv + D(u^2 + v^2),$$

which is also an equation of a circle or a straight line. This proves the theorem. □

Suppose that a generalized circle C divides $\widehat{\mathbb{C}}$ into two generalized discs, say U_1 and U_2. If f is a Möbius transformation, and maps C to another generalized circle C', which divides $\widehat{\mathbb{C}}$ into U_1' and U_2'. Then either maps $f(U_1) = U_1'$ and $f(U_2) = U_2'$, or $f(U_1) = U_2'$ and $f(U_2) = U_1'$. If for any $z_0 \in U_1$, $f(z_0) \in U_1'$, then the first case happens; otherwise the second case happens. On the other hand, if f maps a generalized disc onto a generalized disc, then it maps the boundary of the first disc onto the boundary of the second disc. For example, $\mathbb{R} \cup \{\infty\}$ is called the extended real line, which is the boundary of $\mathbb{H}$ in $\widehat{\mathbb{C}}$. If $f \in \mathrm{Aut}(\mathbb{H})$ is a Möbius transformation, then f maps $\widehat{\mathbb{R}}$ onto $\widehat{\mathbb{R}}$.

Theorem 5.5.4. *Given any three distinct points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$, and any three distinct points $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$, there exists a unique Möbius transformation f such that $f(z_j) = w_j$, $j = 1, 2, 3$.*

Proof. For the existence, it suffices to show that such f exists if $w_1 = 0$, $w_2 = \infty$, and $w_3 = 1$. In fact, if we let F_{z_1,z_2,z_3} denote a Möbius transformation that maps z_1 to 0, z_2 to ∞, and z_3 to 1, then $F^{-1}_{w_1,w_2,w_3} \circ F_{z_1,z_2,z_3}$ is a Möbius transformation that maps z_j to w_j, $j = 1, 2, 3$.

Now we show that F_{z_1,z_2,z_3} exists. First suppose $z_1, z_2, z_3 \in \mathbb{C}$. The map $z \mapsto \frac{z-z_1}{z-z_2}$ takes z_1 to 0 and z_2 to ∞. But it may not take z_3 to 1. To solve this, we may multiply by a suitable constant. So we may construct F_{z_1,z_2,z_3} by

$$F_{z_1,z_2,z_3}(z) = \frac{z_3 - z_2}{z_3 - z_1} \cdot \frac{z - z_1}{z - z_2}.$$

If $\infty \in \{z_1, z_2, z_3\}$, we let

$$F_{z_1,z_2,z_3}(z) = \frac{z_3 - z_2}{z - z_2}, \quad z_1 = \infty;$$

$$F_{z_1,z_2,z_3}(z) = \frac{z - z_1}{z_3 - z_1}, \quad z_2 = \infty;$$

$$F_{z_1,z_2,z_3}(z) = \frac{z - z_1}{z - z_2}, \quad z_3 = \infty.$$

This finishes the proof of the existence part.

Now we show the uniqueness. If f and g both satisfies the property, then $g^{-1} \circ f$ is a Möbius transformation with three fixed points z_1, z_2, z_3. So $h := F_{z_1,z_2,z_3} \circ g^{-1} \circ f \circ F^{-1}_{z_1,z_2,z_3}$ is a Möbius transformation with fixed points $0, 1, \infty$. Since $h(\infty) = \infty$, h has the form $h(z) = az + b$. Since $h(0) = 0$, $b = 0$. Since $h(1) = 1$, $a = 1$. So $h = \mathrm{id}$, which implies that $g = f$. □

Given three distinct points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$, and three distinct points $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$, we will use the following methods to find the Möbius transformation F, which maps z_j to w_j, $j = 1, 2, 3$.

1. $F = F^{-1}_{w_1,w_2,w_3} \circ F_{z_1,z_2,z_3}$, where F_{z_1,z_2,z_3} and F_{w_1,w_2,w_3} are given by the above theorem. If you use this method, you should simplify your answer as much as possible.

2. Write $F(z) = \frac{az+b}{cz+d}$, and solve the equations $\frac{az_j+b}{cz_j+d} = w_j$, $j = 1, 2, 3$, to get a, b, c, d. Note that all equations are linear. There are essentially only 3 unknown variables because

multiplying any nonzero complex numbers to a, b, c, d does not change the transformation. You may assume one variable, say a, equals 1. Sometimes, this does not work if a turns out to be 0. If that is the case, you may then set another variable, say c to be 1.

Homework. Chapter VII, §5: 3(a,b,c), 11.
Additional:

1. Show that if $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$, then $f(z) = \frac{az+b}{cz+d} \in \mathrm{Aut}(\mathbb{H})$.

Definition 5.5.3. *Let z_1, z_2, z_3, z_4 be distinct points in $\widehat{\mathbb{C}}$. Define their cross ratio to be*

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_2 - z_4}{z_1 - z_4} = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4},$$

if $z_1, z_2, z_3, z_4 \in \mathbb{C}$. If any z_j is ∞, then $[z_1, z_2, z_3, z_4]$ is defined by the above formula without the two factors involving z_j. For example, $[z_1, z_2, z_3, \infty] = \frac{z_1 - z_3}{z_2 - z_3}$.

The cross ratio satisfies the following symmetry relations:

$$[z_2, z_1, z_3, z_4] = [z_1, z_2, z_4, z_3] = [z_1, z_2, z_3, z_4]^{-1};$$

$$[z_3, z_4, z_1, z_2] = [z_1, z_2, z_3, z_4].$$

Theorem 5.5.5. *Let F be a Möbius transformation, and $z_j' = F(z_j)$, $j = 1, 2, 3, 4$. Then $[z_1', z_2', z_3', z_4'] = [z_1, z_2, z_3, z_4]$.*

Proof. One may check that $[z_1, z_2, z_3, z_4] = F_{z_2, z_1, z_3}(z_4)$, where F_{z_2, z_1, z_3} is the Möbius transformation that maps z_2, z_1, z_3 to $0, \infty, 1$. Note that $F_{z_2, z_1, z_3} = F_{z_2', z_1', z_3'} \circ F$. Thus,

$$[z_1, z_2, z_3, z_4] = F_{z_2', z_1', z_3'} \circ F(z_4) = F_{z_2', z_1', z_3'}(z_4') = [z_1', z_2', z_3', z_4'].$$

□

Remark. From $[z_1, z_2, z_3, z_4] = F_{z_2, z_1, z_3}(z_4)$ and that $z_4 \neq z_j$, $j = 1, 2, 3$, we see that $[z_1, z_2, z_3, z_4] \notin \{0, \infty, 1\}$.

Homework.
Show that the four distinct points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ lie on a generalized circle if and only if $[z_1, z_2, z_3, z_4] \in \mathbb{R}$.

5.6 Riemann's Mapping Theorem

Let D_1 and D_2 be two complex domains. We say that D_1 is conformally equivalent to D_2, and write $D_1 \cong D_2$ if $\mathrm{Iso}(D_1, D_2)$ is not empty, i.e., there is an analytic isomorphism of D_1 with D_2. From a homework problem, "$\cong$" is an equivalence relation: $D \cong D$; $D_1 \cong D_2$ implies $D_2 \cong D_1$; $D_1 \cong D_2$ and $D_2 \cong D_3$ imply $D_1 \cong D_3$.

Theorem 5.6.1. **[Riemann Mapping Theorem]** *Let $U \subsetneqq \mathbb{C}$ be a simply connected domain. Let $z_0 \in U$. Then there exists $f \in \mathrm{Iso}(U, \mathbb{D})$ such that $f(z_0) = 0$. Moreover, such f is unique up to a rotation, i.e., if g also satisfies the property of f, then $g = M_c \circ f$ for some $c \in \mathbb{C}$ with $|c| = 1$. If we require that $f'(z_0) > 0$, then f is unique.*

The above theorem implies that every simply connected complex domain other than $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$. Here the case $U = \mathbb{C}$ is excluded because $\mathrm{Iso}(\mathbb{C}, \mathbb{D})$ is empty by Liouville. The theorem is useful for two reasons. First, it transforms results about the unit disc into those about any simply connected domain. Second, the proof of the theorem introduces some important ideas.

Homework. Let $U \subsetneqq \mathbb{C}$ be a simply connected domain, and $f : U \to U$ be analytic. Suppose that f fixes at least two points in U, i.e., there are $z_1 \neq z_2 \in U$ such that $f(z_j) = z_j$, $j = 1, 2$. Prove that f is identity.

5.7 Limits of Sequence of Analytic Functions

Definition 5.7.1. *A sequence of functions (f_n) on an open set U is said to converge compactly to a function f in U, if for every compact set $K \subset U$, $f_n \to f$ uniformly on K.*

Remarks.

1. Using the open covering definition of compact sets, one can show that $f_n \to f$ compactly in U is equivalent to that $f_n \to f$ locally uniformly in U, i.e., for every $z_0 \in U$, there is $r > 0$ such that $f_n \to f$ uniformly in $D(z_0, r)$.

2. If every f_n is continuous, then the compactly convergent limit f is also continuous.

Theorem 5.7.1. *Let (f_n) be a sequence of analytic functions on an open set U, which converges compactly to f. Then f is analytic in U, and (f_n') converges compactly to f' in U.*

Proof. Let $z_0 \in U$. Pick $r > 0$ such that $D(z_0, r) \subset U$. Let γ be any closed curve in $D(z_0, r)$. Since each f_n is analytic analytic in $D(z_0, r)$, we have $\int_\gamma f_n = 0$. Since γ is compact, we have $f_n \to f$ uniformly on γ, $\int_\gamma f = \lim_{n\to\infty} \int_\gamma f_n = 0$. From Morera's Theorem, f is holomorphic on $D(z_0, r)$. Especially, f is complex differentiable at z_0. Since $z_0 \in U$ is arbitrary, f is holomorphic on U.

To prove that (f_n') converges compactly to f' in U, it suffices to show that, if $\bar{D}(z_0, r) \subset U$, then $f_n' \to f'$ uniformly on $\bar{D}(z_0, r)$. We may choose $R > r$ such that $\bar{D}(z_0, R) \subset U$. Let $J = \{|z - z_0| = R\}$. From Cauchy's Formula, for any $z \in D(z_0, R)$,

$$f_n'(z) - f'(z) = \frac{1}{2\pi i} \int_J \frac{f_n(w) - f(w)}{(w - z)^2}\, dw,$$

which implies that

$$|f_n'(z) - f'(z)| \le \frac{1}{2\pi} L(J) \frac{\|f_n - f\|_J}{d(z, J)^2}.$$

Thus,

$$\|f_n' - f'\|_{\bar{D}(z_0,r)} \le \frac{R\|f_n - f\|_J}{\inf_{z\in\bar{D}(z_0,r)} d(z,J)^2} = \frac{R\|f_n - f\|_J}{(R-r)^2},$$

which tends to 0 because $f_n \to f$ uniformly on the compact set J. Thus, $f_n' \to f'$ uniformly on $\bar{D}(z_0, r)$. □

Definition 5.7.2. *We say that a series of functions $\sum_{n=1}^\infty f_n$ converges compactly to f in U, if the partial sum sequence $s_n = \sum_{k=1}^n f_k$ converges to f compactly in U, i.e., $\sum_{n=1}^\infty f_n$ converges uniformly on every compact subset of U.*

From the previous theorem, if every f_n is analytic in U, then so is the compactly convergent sum $f = \sum f_n$. Moreover, the series $\sum f_n'$ converges compactly to f' in U. Recall the comparison principle: given $K \subset U$, if there is a sequence (c_n) depending on K such that $\|f_n\|_K \le c_n$ for each n, and $\sum c_n < \infty$, then $\sum f_n$ converges uniformly on K.

Example.

1. Suppose the power series $\sum_{k=0}^\infty a_k z^k$ has radius $R > 0$. Then the series converges compactly in $D(0, R)$. In fact, we know that, for every $r \in (0, R)$, the series converges uniformly on $\bar{D}(0, r)$. So the compact convergence follows from the fact that, for every compact $K \subset D(0, R)$, there exists $r \in (0, R)$ such that $K \subset \bar{D}(0, r)$.

2. Consider the series
$$f(z) = \sum_{n=1}^\infty \frac{1}{n^z},$$
where n^z is understood as the analytic function $e^{z \log n}$. The $\log n$ is the real logarithm function. We do not need to consider the branch of $\log z$. We have
$$\left|\frac{1}{n^z}\right| = |e^{-z\log n}| = e^{-\operatorname{Re} z \log n} = \frac{1}{n^{\operatorname{Re} z}}.$$
We know that, for every $p > 1$,
$$\sum_{n=1}^\infty \frac{1}{n^p} < \infty.$$
By comparison principle, for any $p > 1$, $\sum_{n=1}^\infty \frac{1}{n^z}$ converges uniformly on $\{z : \operatorname{Re} z \ge p\}$. Since for every compact set $K \subset \{\operatorname{Re} z > 1\}$, there is $p > 1$ such that $K \subset \{\operatorname{Re} z \ge p\}$, we see that $\sum_{n=1}^\infty \frac{1}{n^z}$ converges compactly in $\{\operatorname{Re} z > 1\}$. From the above theorem, the sum f is analytic in $\{\operatorname{Re} z > 1\}$, and
$$f'(z) = \sum_{n=1}^\infty \frac{-\log n}{n^z}.$$
Such f has an analytic extension to $\mathbb{C} \setminus \{1\}$, which is the famous Riemann zeta function. It is often denoted by $\zeta(z)$. It has trivial zeros at negative integers: $-2, -4, -6, \dots$. The Riemann hypothesis states that, all nontrivial zeros of ζ lie on the vertical line $\{\operatorname{Re} z = \frac{1}{2}\}$.

3. Let $a < b \in \mathbb{R}$ and $U \subset \mathbb{C}$ be open. Suppose $f : [a, b] \times U \to \mathbb{C}$ be continuous, and for every $t \in [a, b]$, $z \mapsto f(t, z)$ is analytic in U. Then $F(z) := \int_a^b f(t, z)dt$ is analytic in U. In fact, if we define the Riemann sum function

$$F_n(z) = \sum_{k=1}^{n} f(t_k, z)(t_k - t_{k-1}), \quad z \in U,$$

where $t_k = a + \frac{k}{n}(b - a)$, then each F_n is analytic, and $F_n \to F$ compactly in U.

4. Suppose f is a continuous function on $\mathbb{R}$. From above, we see that for any $n \in \mathbb{N}$, $F_n(z) = \int_{-n}^{n} e^{itz} f(t)dt$ is analytic in $\mathbb{C}$. Recall that the Fourier transformation of f is $\widehat{f}(z) = \int_{-\infty}^{\infty} e^{itz} f(t)dt$. If there is an open set U such that (F_n) converges compactly in U, then the Fourier transformation $\widehat{f}$ is analytic in U.

5. The Gamma function is defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx = \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \int_\varepsilon^R x^{t-1} e^{-x}.$$

One can show that the limit converges compactly in the right half plane $\mathbb{H}_R = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. So this formula defines an analytic function on $\mathbb{H}_R$. It can be extended to an analytic function on $\mathbb{C} \setminus \{n \in \mathbb{Z} : n \le 0\}$. The most important property of Γ is that it is an analytic extension of the factorial function:

$$\Gamma(n) = (n - 1)!$$

In summary, we have the following methods to construct/check new analytic functions, which are helpful since analytic functions satisfy a lot of interesting properties.

1. Definition of the complex derivative
2. Combination of known analytic functions: $f \pm g$, $f \cdot g$, f/g, $f \circ g$
3. Cauchy-Riemann equations
4. Power series and Laurent series
5. Primitive or local primitive of an analytic function
6. Derivative of an analytic function
7. Inverse or local inverse of an analytic function
8. Limit of a compactly convergent sequence or series of analytic functions
9. Integral of a family of analytic functions

Homework Chapter V §1: 2, 3 (a,c);

5.8 Normal Families

Definition 5.8.1. *Let U be an open set. Let Φ be a family of analytic functions defined on U. We say that Φ is a normal family, if every sequence in Φ contains a subsequence, which converges compactly in U. The limit function does not have to belong to Φ.*

Remark. Let Σ denote the set of analytic functions on U. It is possible to define a metric d on Σ such that $d(f_n, f) \to 0$ iff $f_n \to f$ compactly in U. More specifically, we may find an increasing sequence of compact subsets (K_n) of U such that K_n is contained in the interior of K_{n+1}, and $U = \bigcup_n K_n$. For example, we may choose

$$K_n = \{z \in U : |z| \le n, \operatorname{dist}(z, U^c) \ge 1/n\}, \quad n \in \mathbb{N}.$$

Then we define

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}}.$$

Then $d(f_n, f) \to 0$ if and only if $\|f_n - f\|_{K_n} \to 0$ for each n, which is further equivalent to that $f_n \to f$ compactly in U since every compact subset of U is contained in one of K_n. Then $\Phi \subset \Sigma$ is a normal family iff Φ is a precompact set with respect to this metric, i.e., the closure of Φ is compact.

We will uses the famous Arzelà-Ascoli theorem, which is stated below.

Theorem 5.8.1. *Let K and L be two compact metric spaces. Let $f_n : K \to L$, $n \in \mathbb{N}$, be an equicontinuous sequence of functions. Then (f_n) contains a subsequence, which converges uniformly on K.*

We say that (f_n) is equicontinuous on K, if for every $\varepsilon > 0$, there is $\delta > 0$ such that for any n and any $z, w \in K$, $d_K(z, w) < \delta$ implies $d_L(f_n(z), f_n(w)) < \varepsilon$.

Now we briefly describe the proof of the Arzelà-Ascoli theorem. Using the compactness of K, one may find a countable dense subset of K: $\{z_m : m \in \mathbb{N}\}$. Let $n_k^0 = k$. Consider the values of $(f_{n_k^0})$ on z_1. Using the compactness of L, we can find a subsequence $(f_{n_k^1})$ of $(f_{n_k^0})$ such that $(f_{n_k^1}(z_1))$ is a convergent sequence in L. Consider the values of $(f_{n_k^1})$ on z_2. There is a subsequence $(f_{n_k^2})$ of $(f_{n_k^1})$ such that $(f_{n_k^2}(z_2))$ is a convergent sequence. Repeating this process, we get a sequence of subsequences $(f_{n_k^m})$, $m \in \mathbb{N}$, such that $(f_{n_k^m})$ is a subsequence of $(f_{n_k^{m-1}})$, and $f_{n_k^m}(z_m)$ converges as $k \to \infty$. Now let $n_k = n_k^k$. Then for every z_m, $(f_{n_k}(z_m))$ converges as $k \to \infty$. The reason is that, for any m, the sequence (n_k) is a subsequence of $(n_k^m : k \in \mathbb{N})$ except for finitely many elements. The method used above is called a diagonal procedure.

Then one may use the equicontinuity of (f_n) and the denseness of (z_m) in K to conclude that the pointwise convergence of (f_{n_k}) on $\{z_m\}$ implies the uniformly Cauchy property of (f_{n_k}) on the whole space K.

Fix $\varepsilon > 0$. Choose $\delta > 0$ such that for any n and any $z, w \in K$, $d_K(z, w) < \delta$ implies $d_L(f_n(z), f_n(w)) < \varepsilon/3$. Since $\{z_m : m \in \mathbb{N}\}$ is dense in K, we may pick finite set $S \subset \{z_m :$

$m \in \mathbb{N}\}$ such that for any $z \in K$, there is $w \in S$ with $|z-w| < \delta$. Since $\lim_{k\to\infty} f_{n_k}(w)$ converges for every $w \in S$, there is N such that $|f_{n_{k_1}}(w) - f_{n_{k_2}}(w)| < \varepsilon/3$ for any $k_1, k_2 \geq N$ and $w \in S$. This implies that, for any $z \in S$ and $k_1, k_2 \geq N$, $|f_{n_{k_1}}(z) - f_{n_{k_2}}(z)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. From this, we say that (f_{n_k}) is uniformly Cauchy on K. So it converges uniformly on K.

Let's see how Arzelà-Ascoli theorem can be applied in the context of Complex Analysis.

Lemma 5.8.1. *Suppose K is a compact subset of an open set $U \subset \mathbb{C}$. Let (f_n) be a sequence of analytic functions on U. If (f_n') or (f_n) is uniformly bounded on U, then (f_n) is equicontinuous on K.*

Proof. First, we assume that (f_n') is uniformly bounded on U. Then there is $M > 0$ such that $|f_n'| \leq M$ on U for any n. We may find $r > 0$ such that $D(z_0, r) \subset U$ for every $z_0 \in K$. This implies that, if $z, w \in K$ and $|z - w| < r$, then $[z, w] \subset U$, and so $|f_n(z) - f_n(w)| = |\int_{[w,z]} f_n'| \leq M|w - z|$. For any $\varepsilon > 0$, let $\delta = \min\{r, \varepsilon/M\} > 0$. Then $z, w \in K$ and $|z - w| < \delta$ implies that $|f_n(z) - f_n(w)| < \varepsilon$ for any n. So (f_n) is equicontinuous on K.

Second, we assume that (f_n) is uniformly bounded on U. Then there is $M \in \mathbb{R}$ with $M > 0$ such that $\|f_n\|_U \leq M$ for every n. Let $r > 0$ be as above. Let $U' = \bigcup_{z\in K} D(z, r/2)$. Then U' is an open set that contains K. Moreover, for any $z_0 \in U'$, $\bar{D}(z_0, r/2) \subset U$. Fix $z_0 \in U'$. From Cauchy's Formula, we get

$$f_n'(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r/2} \frac{f_n(z)}{(z - z_0)^2} dz, \quad n \in \mathbb{N},$$

which implies that

$$|f_n'(z_0)| \leq \frac{1}{2\pi} \frac{\|f_n\|_U}{(r/2)^2} L(\{|z - z_0| = r/2\}) \leq \frac{2M}{r}, \quad n \in \mathbb{N}.$$

This implies that (f_n') is uniformly bounded on U'. Since K is a compact subset of U', from the result of the last paragraph, we see that (f_n) is equicontinuous on K. $\square$

Corollary 5.8.1. *Suppose K is a compact subset of an open set $U \subset \mathbb{C}$. Let (f_n) be a uniformly bounded sequence of analytic functions on U. Then (f_n) contains a subsequence, which converges uniformly on K.*

Proof. From the previous lemma, (f_n) is equicontinuous on K. Since (f_n) is uniformly bounded on K, there is M such that $\|f_n\|_K \leq M$ for any n. Let $L = \overline{D}(0, M)$. Then L is a compact set, and f_n maps K into L. The A-A Theorem implies the result. $\square$

Theorem 5.8.2. [Montel's Theorem] *Let U be an open set. Let Φ be a family of analytic functions on U. Then Φ is a normal family if and only if Φ is uniformly bounded on every compact subset of U.*

Proof. The "only if" part holds because if Φ is not uniformly bounded on a compact $K \subset U$, then we may find a sequence (f_n) from Φ such that $\|f_n\|_K \to \infty$, which can not contain a subsequence that converges uniformly on K.

Now we prove the "if" part. Suppose Φ is uniformly bounded on every compact subset of U. For $m \in \mathbb{N}$, let $K_m = \{z \in U : \operatorname{dist}(z, U^c) \geq 1/m, |z| \leq m\}$. Then each K_m is a compact subset of U. Let $U_m = \bigcup_{z \in K_m} D(z, \frac{1}{m} - \frac{1}{m+1})$, $m \in \mathbb{N}$. Then each U_m is an open set, $K_m \subset U_m \subset K_{m+1}$, and $U = \bigcup U_m$.

Let (f_n) be a sequence in Φ. From the assumption on Φ, (f_n) is uniformly bounded on each K_m. Since $U_m \subset K_{m+1}$, (f_n) is uniformly bounded on each U_m. For each $m \in \mathbb{N}$, since K_m is a compact subset of the open set U_m, from the above corollary, (f_n) contains a subsequence that converges uniformly on K_m.

We now use another diagonal procedure to conclude that (f_n) contains a subsequence, which converges uniformly on every K_m. Let $n_k^0 = k$. Then $(f_{n_k^0})$ contains a subsequence $(f_{n_k^1})$, which converges uniformly on K_1. And $(f_{n_k^1})$ contains a subsequence $(f_{n_k^2})$, which converges uniformly on K_2. Repeating this process, we get a sequence of subsequences $(f_{n_k^m})$, $m \in \mathbb{N}$, such that $(f_{n_k^m})$ is a subsequence of $(f_{n_k^{m-1}})$, and $(f_{n_k^m})$ converges uniformly on K_m. Let $n_k = n_k^k$. Then (f_{n_k}) converges uniformly on every K_m.

Finally, if $K \subset U$ is a compact set, then there exists $R, r > 0$ such that $K \subset D(0, R)$ and for every $z \in K$, $D(z, r) \subset U$. If $m \in \mathbb{N}$ satisfies $m > \max\{R, 1/r\}$, then $K \subset K_m$, which implies that (f_{n_k}) converges uniformly on K. Thus, (f_{n_k}) converges compactly in U. $\square$

Remark. That Φ is uniformly bounded on every compact subset of U is equivalent to that Φ is locally uniformly bounded in U, i.e., for every $z_0 \in U$, there is $r > 0$ such that Φ is uniformly bounded on $\bar{D}(z_0, r)$.

Homework. Chapter X, §2: 6, 7.

1. Let Φ be the set of analytic functions on an open set U which satisfies that, for all $f \in \Phi$, $\int_U |f(x,y)| dx dy \leq 1$. Prove that Φ is a normal family.

5.9 Proof of the Riemann Mapping Theorem

We first prove the uniqueness part. Suppose f_1 and f_2 are analytic isomorphisms of U with $\mathbb{D}$ such that $f_j(z_0) = 0$, $j = 1, 2$. Let $f = f_1 \circ f_2^{-1}$. Then $f \in \operatorname{Aut}(\mathbb{D})$ and $f(0) = 0$. So $f = M_c$ for some $c \in \mathbb{C}$ with $|c| = 1$, which implies $f_1 = M_c \circ f_2$. If $f_1'(z_0)$ and $f_2'(z_0)$ are both positive, then so is $f'(0) = f_1'(z_0)/f_2'(z_0)$, which implies that $c = 1$ and $f_1 = f_2$.

Now we prove the existence part. Let Φ denote the set of injective analytic $f : U \to \mathbb{D}$ such that $f(z_0) = 0$. First, we show that Φ is not empty. Let $c \in \mathbb{C} \setminus U$. Since U is simply connected, there is a branch $L(z)$ of $\log(z - c)$. Such L is an analytic isomorphism of U. Let $V = L(U)$. Then V is open, and $V \cap V + (2\pi i) = \emptyset$. Let $w_0 \in V + 2\pi i$. Then there is $r > 0$ such that $D(w_0, r) \subset V + 2\pi i$, which implies that $D(w_0, r) \cap V = \emptyset$. Thus, $h(z) := \frac{r}{L(z) - w_0}$ is an analytic isomorphism of U such that $h(U) \subset \mathbb{D}$. Finally, let $f = g_{h(z_0)} \circ h$. Then $f \in \Phi$.

Second, we observe that, if $f_0 \in \Phi \cap \mathrm{Iso}(U, \mathbb{D})$, then for any $f \in \Phi$, $F := f \circ f_0^{-1}$ is an analytic map from $\mathbb{D}$ into $\mathbb{D}$ that satisfies $F(0) = 0$. From Schwarz lemma, we get $|F'(0)| \le 1$, which implies that $|f'(z_0)| \le |f_0'(z_0)|$. Thus, if such f_0 exists, then $|f_0'(z_0)| = \max\{|f'(z_0)| : f \in \Phi\}$.

Third, let $S = \sup\{|f'(z_0)| : f \in \Phi\}$. Then we may pick a sequence (f_n) from Φ such that $|f_n'(z_0)| \to S$. Since Φ is uniformly bounded on U, it is a normal family. Thus, (f_n) contains a subsequence (f_{n_k}), which converges compactly in U. Let f_0 be the limit. Then f_0 is analytic in U, and $f_{n_k}' \to f_0'$ compactly in U. In particular, we see that $f_0(z_0) = \lim f_{n_k}(z_0) = 0$ and $|f_0'(z_0)| = \lim |f_{n_k}'(z_0)| = S > 0$. Thus, f_0 is not constant.

Since $|f_{n_k}(z)| < 1$ on U for each k, and $f_0 = \lim f_{n_k}$, we get $|f_0| \le 1$ on U. If there is some $z_0 \in U$ such that $|f_0(z_0)| = 1$, then from the Maximum modulus principle, we can conclude that f_0 is constant, which is impossible. Thus, $f_0 : U \to D$.

Now we show that f_0 is an analytic isomorphism of U. We have that $f_{n_k} \to f_0$ compactly in U, each f_{n_k} is analytic and injective, and f_0 is not constant. The argument below will show that, if a sequence of injective analytic functions (f_{n_k}) converges compactly to f_0, which is not constant, then f_0 is also injective.

Suppose f_0 is not injective. Then there exist $z_1 \ne z_2 \in U$ such that $f_0(z_1) = f_0(z_2) = w_0$. In other words, z_1 and z_2 are zeros of $f_0 - w_0$. Since f_0 is not constant, $f_0 - w_0$ is not constant zero. We may find $r > 0$ such that $\bar{D}(z_1, r)$ and $\bar{D}(z_2, r)$ are disjoint subsets of U, and $f_0 - w_0 \ne 0$ on $C_j := \{|z - z_j| = r\}$, $j = 1, 2$. Since $C_1 \cup C_2$ is compact, and $f_0 - w_0 \ne 0$ on $C_1 \cup C_2$, there is $\varepsilon > 0$ such that $|f_0(z) - w_0| \ge \varepsilon$, $z \in C_j$, $j = 1, 2$. Since (f_{n_k}) converges to f_0 uniformly on $C_1 \cup C_2$, there is N such that when $k > N$, $\|f_{n_k} - f_0\|_{C_1 \cup C_2} < \varepsilon$. Since

$$|(f_{n_k}(z) - w_0) - (f_0(z) - w_0)| \le \|f_{n_k} - f_0\|_{C_1 \cup C_2} < \varepsilon \le |f_0(z) - w_0|, \quad z \in C_1 \cup C_2,$$

from Rouché's theorem and the fact that $f_0 - w_0$ has zeros in both $D(z_1, r)$ and $D(z_2, r)$, we see that, if $k > N$, then $f_{n_k} - w_0$ has zeros in both $D(z_1, r)$ and $D(z_2, r)$, which contradicts that f_{n_k} is injective. Thus, f_0 is injective.

Now we see that $f_0 \in \Phi$ and $|f'(z_0)| = S = \sup\{|f'(z_0)| : f \in \Phi\}$. So $|f'(z_0)| = \max\{|f'(z_0)| : f \in \Phi\}$. To complete the proof, it remains to show that $f_0(U) = \mathbb{D}$. Suppose $f_0(U) \ne \mathbb{D}$. Let $a \in \mathbb{D} \setminus f_0(U)$ and $h_1 = g_a \circ f$. Recall that $g_a(z) = \frac{a-z}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{D})$ is such that $g_a \circ g_a = \mathrm{id}$, $g_a(a) = 0$, and $g_a(0) = a$. Then h_1 is injective and analytic, and has no zeros on U. Since U is simply connected, there is an analytic branch h_2 of $h_1^{1/2}$ in U. Since $h_1 = h_2^2$, we see that h_2 is also injective. Since $|h_2(z)| = |h_1(z)|^{1/2} < 1$ for $z \in U$, h_2 maps U into $\mathbb{D}$. Let $b = h_2(z_0) \in \mathbb{D}$. Then

$$b^2 = h_2(z_0)^2 = h_1(z_0) = g_a(f_0(z_0)) = g_a(0) = a.$$

Let $f_1 = g_b \circ h_2$. Then $f_1 : U \to \mathbb{D}$ is analytic and injective and $f_1(z_0) = g_b(h_2(z_0)) = g_b(b) = 0$. Thus, $f_1 \in \Phi$. Let $S(z) = z^2$. Then

$$f_0 = g_a \circ h_1 = g_a \circ S \circ h_2 = g_a \circ S \circ g_b \circ f_1.$$

We see that $f_1(z_0) = 0$, $g_b(0) = b$, $S(b) = a$, and $g_a(a) = 0$. Thus, $\frac{|f_0'(z_0)|}{|f_1'(z_0)|} = |g_a'(a)||S'(b)||g_b'(0)|$. It is straightforward to check that $g_c'(0) = |c|^2 - 1$ and $g_c'(c) = \frac{1}{|c|^2-1}$. Thus,

$$\frac{|f_0'(z_0)|}{|f_1'(z_0)|} = \frac{2|b|(1-|b|^2)}{1-|a|^2} = \frac{2|b|(1-|b|^2)}{1-|b|^4} = \frac{2|b|}{1+|b|^2} < 1,$$

which contradicts that $|f_0'(z_0)|$ maximizes $\{|f'(z_0)| : f \in \Phi\}$. So $f_0 \in \mathrm{Iso}(U, \mathbb{D})$ is what we need.

5.10 Examples

We will see some examples of simply connected domains, for which the analytic isomorphisms between these domains and the unit disc or half plane can be explicitly expressed.

Examples.

1. Let S denote the square map $z \mapsto z^2$. Recall that $|S(z)| = |z|^2$ and $\arg S(z) = 2\arg(z)$. Since $S(z_1) = S(z_2)$ if and only if $z_1 = z_2$ or $z_2 = -z_2$, we see that S is an analytic isomorphism of U if and only if $U \cap (-U) = \emptyset$.

2. Recall that $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\} = \{z \in \mathbb{C} : 0 < \arg(z) < \pi\}$. Let $\mathbb{H}_R = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\} = \{z \in \mathbb{C} : -\pi/2 < \arg(z) < \arg(z)\}$ denote the right half plane. We have $S \in \mathrm{Iso}(\mathbb{H}, \mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\})$, $S \in \mathrm{Iso}(\mathbb{H}_R, \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\})$, and $M_i \in \mathrm{Iso}(\mathbb{H}_R, \mathbb{H})$, where $M_i(z) = iz$.

3. The intersection $\mathbb{H}_R \cap \mathbb{H} = \{z \in \mathbb{C} : 0 < \arg(z) < \pi/2\}$ is the first quadrant. Since S doubles the argument, we have $S \in \mathrm{Iso}(\mathbb{H}_R \cap \mathbb{H}, \mathbb{H})$.

4. Let $U = \mathbb{D} \cap \mathbb{H}_R \cap \mathbb{H}$ be a quarter disc. Then $S \in \mathrm{Iso}(U, \mathbb{H} \cap \mathbb{D})$. This follows from the previous example and the fact that $S(z) \in \mathbb{D}$ iff $z \in \mathbb{D}$.

5. Let $F(z) = \frac{z+1}{-z+1}$ be a Möbius transformation. First, note that for every $x \in \widehat{\mathbb{R}}$, $F(x) \in \widehat{\mathbb{R}}$. Thus, $F(\mathbb{H}) = \mathbb{H}$ or $F(\mathbb{H}) = -\mathbb{H}$. Since $F(i) = i$, we get $F \in \mathrm{Iso}(\mathbb{H}, \mathbb{H})$. Second, note that $F^{-1}(z) = \frac{z-1}{z+1}$, so $F^{-1} \circ M_i^{-1}(z) = \frac{-iz-1}{-iz+1} = \frac{z-i}{z+i} \in \mathrm{Iso}(\mathbb{H}, \mathbb{D})$, which implies that $M_i \circ F \in \mathrm{Iso}(\mathbb{D}, \mathbb{H})$. Since $M_i \in \mathrm{Iso}(\mathbb{H}_R, \mathbb{H})$, we see that $F \in \mathrm{Iso}(\mathbb{D}, \mathbb{H}_R)$. Combining these two facts, we get $F \in \mathrm{Iso}(\mathbb{H} \cap \mathbb{D}, \mathbb{H} \cap \mathbb{H}_R)$.

6. Let S and F be as above. Then $G := S \circ F \circ S \in \mathrm{Iso}(\mathbb{D} \cap \mathbb{H}_R \cap \mathbb{H}, \mathbb{H})$.

7. The map $z \mapsto \sqrt{z^2 - 1}$ is an isomorphism of $\mathbb{H}_R \setminus [0, 1]$ with $\mathbb{H}_R$. To see this, first, $z \mapsto z^2$ maps $\mathbb{H}_R \setminus [0, 1]$ onto $\mathbb{C} \setminus (-\infty, 1]$. Then $z \mapsto z - 1$ maps $\mathbb{C} \setminus (-\infty, 1]$ onto $\mathbb{C} \setminus (-\infty, 0]$. Finally, $z \mapsto \sqrt{z}$ maps $\mathbb{C} \setminus (-\infty, 0]$ onto $\mathbb{H}_R$.

8. The exponential map exp is an analytic isomorphism of U if and only if $U \cap (2k\pi i + U) = \emptyset$ for any $k \in \mathbb{Z}$.

9. For every $y > 0$, let $\mathbb{S}_y$ denote the strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < y\}$. The map exp is an analytic isomorphism of $\mathbb{S}_\pi$ with $\mathbb{H}$, an analytic isomorphism of the half strip $\mathbb{S}_\pi \cap (-\mathbb{H}_R)$ with the half disc $\mathbb{D} \cap \mathbb{H}$, and the strip $\mathbb{S}_{2\pi}$ with the plane without a half line: $\mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\}$.

10. Let $J(z) = 1/z$. Then $J \in \operatorname{Aut}(\mathbb{C} \setminus \{0\})$ and $J^{-1} = J$. Moreover, $J \in \operatorname{Iso}(\mathbb{D}, \mathbb{C} \setminus \overline{\mathbb{D}})$ and $J \in \operatorname{Iso}(\mathbb{H}, -\mathbb{H})$. So $J \in \operatorname{Iso}(\mathbb{H} \setminus \overline{\mathbb{D}}, (-\mathbb{H}) \cap \mathbb{D})$.

11. Let $f(z) = z + 1/z$. Note that $f(z_1) = f(z_2)$ if and only if $z_1 = z_2$ or $z_1 = J(z_2)$. So f is an analytic isomorphism of U if and only if $U \cap J(U) = \emptyset$. We claim that $f(\mathbb{H}) = \mathbb{C} \setminus ((-\infty, 2] \cup [2, \infty))$. To see this, we observe that for any $w \in \mathbb{C}$, $f(z) = w$ is equivalent to the equation $z^2 - wz + 1 = 0$, which has two roots z_1, z_2 that satisfy $z_1 z_2 = 1$ (the two roots coincide, i.e., $z_1 = z_2$ when $w = \pm 2$). From this, we get $f(\mathbb{C} \setminus \{0\}) = \mathbb{C}$. Also observe that $f(z) \in (-\infty, -2] \cup [2, \infty)$ iff $z \in \mathbb{R} \setminus \{0\}$. Thus, $f(\mathbb{H} \cup (-\mathbb{H})) = \mathbb{C} \setminus ((-\infty, 2] \cup [2, \infty))$. Since $f \circ J = f$, we get $f(\mathbb{H}) = f(\mathbb{H} \cup J(\mathbb{H})) = f(\mathbb{H} \cup (-\mathbb{H}))$. Since $J(\mathbb{H}) = -\mathbb{H}$ is disjoint from $\mathbb{H}$, we get $f \in \operatorname{Iso}(\mathbb{H}, \mathbb{C} \setminus ((-\infty, 2] \cup [2, \infty)))$.

12. Next, for the f above, we show that $f(\mathbb{H} \setminus \overline{\mathbb{D}}) = \mathbb{H}$. Suppose $z \in \mathbb{H} \setminus \overline{\mathbb{D}}$. Write $z = re^{i\theta}$ with $r > 1$ and $0 < \theta < \pi$. Then

$$f(z) = re^{i\theta} + r^{-1}e^{-i\theta} = (r + r^{-1}) \cos\theta + i(r - r^{-1}) \sin\theta \in \mathbb{H},$$

because $r - r^{-1} > 0$ and $\sin\theta > 0$. On the other hand, if $z \in \mathbb{H}$, and $f(z) \in \mathbb{H}$, then we must have $r = |z| > 1$. So we get $f \in \operatorname{Iso}(\mathbb{H} \setminus \overline{\mathbb{D}}, \mathbb{H})$. Since $f \circ J = f$ and $J \in \operatorname{Iso}(\mathbb{D} \cap (-\mathbb{H}), \mathbb{H} \setminus \overline{\mathbb{D}})$, we get $f \in \operatorname{Iso}(\mathbb{D} \cap (-\mathbb{H}), \mathbb{H})$. Since $f(-z) = -f(z)$, we see that $f \in \operatorname{Iso}(\mathbb{D} \cap \mathbb{H}, -\mathbb{H})$ and $f \in \operatorname{Iso}(-\mathbb{H} \setminus \overline{\mathbb{D}}, -\mathbb{H})$. Finally, f is injective on $\mathbb{C} \setminus \overline{\mathbb{D}}$ since $J(\mathbb{C} \setminus \overline{\mathbb{D}}) = \mathbb{D} \setminus \{0\}$ is disjoint from $\mathbb{C} \setminus \overline{\mathbb{D}}$. We have seen that f maps $\mathbb{H} \setminus \overline{\mathbb{D}}$ onto $\mathbb{H}$, and maps $-\mathbb{H} \setminus \overline{\mathbb{D}}$ onto $-\mathbb{H}$. We also observe that f maps $(1, \infty)$ onto $(2, \infty)$, and maps $(-\infty, -1)$ onto $(-\infty, -2)$. Combining, we see that $f \in \operatorname{Iso}(\mathbb{C} \setminus \overline{\mathbb{D}}, \mathbb{C} \setminus [-2, 2])$.

13. Let W be the half strip $\{z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}, \operatorname{Im} y > 0\}$. Consider the function

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2}\Big(\frac{e^{iz}}{i} + \Big(\frac{e^{iz}}{i}\Big)^{-1}\Big).$$

We see that $\sin z = \frac{1}{2} f \circ g(z)$, where $f(z) = z + z^{-1}$ and $g(z) = e^{iz}/i = e^{i(z - \pi/2)}$. Note that $g \in \operatorname{Iso}(W, \mathbb{D} \cap (-\mathbb{H}))$. From the above example, $f \in \operatorname{Iso}(\mathbb{D} \cap (-\mathbb{H}), \mathbb{H})$. Thus, $\sin z \in \operatorname{Iso}(W, \mathbb{H})$.

Homework.

1. Construct an analytic isomorphism of $\mathbb{D} \setminus [1/2, 1]$ with $\mathbb{H}$.

2. Prove that $f(z) = \frac{z}{(1-z)^2}$ (called the Koebe function) is an analytic isomorphism of $\mathbb{D}$, and find $f(\mathbb{D})$. Hint: Express f as a composition of analytic isomorphisms.

List of topics:

1. Basic computation of complex numbers
2. Triangle inequality
3. Polar form and rectangular form
4. Compute powers and n-th roots using the polar form
5. Complex exponential, logarithm, trigonometric functions and hyperbolic functions
6. Principal logarithm, branch of logarithm, and primitive of $\frac{1}{z}$.
7. Complex powers
8. Topology on $\mathbb{C}$
9. Radius of convergence
10. Cauchy Riemann equations
11. Derivative rules
12. derivatives/antiderivative of power series, and the related differential equations
13. Coefficients of a power series expansion expressed in terms of derivatives
14. Compute the integral over a curve using the definition or the primitive
15. Uniqueness theorem
16. Cauchy's theorem and Cauchy's formula
17. Liouville's theorem
18. Fundamental Theorem of Algebra
19. Properties of simply connected domains: existence of primitive, branch of logarithm, harmonic conjugate.
20. Harmonic function and harmonic conjugate
21. Mean value theorem for analytic functions and harmonic functions
22. Maximum principle for analytic functions and harmonic functions
23. Winding numbers and the general Cauchy's Theorem/Formula and Residue Formula
24. Laurent series: formulas for the coefficients

25. Find singularities, determine the types, and find the orders of poles
26. Behavior near a singularity of different types
27. Laurent series of function $\frac{1}{z-z_0}$ in different annuli
28. Compute the residue
29. Residue formula
30. Rouche's theorem
31. Open Mapping Theorem and Inverse Mapping Theorem
32. Definite integral: half disc, half disc minus a small half disc, rectangular contour, trigonometric integrals, involving branch of logarithm
33. Compute power series or Laurent series of the product or ratio of two power series.
34. Schwarz lemma
35. Find Möbius transformation that takes 3 points to 3 points
36. Describe the image of a circle, line, disc, or half plane under a Möbius transformation
37. Cross ratio
38. Riemann mapping theorem
39. Compactly convergence
40. Normal family
41. Find analytic isomorphisms between some particular domains.