Section 4.2   Linear transformations, their matrices and their geometry

These notes cover and extend Section 4.2 in the textbook. The purpose is to illustrate two principles:

Key principles: for a linear transformation $L : V \to W$

- L is determined by what it does to the basis elements.

- After we choose a basis for V and a basis for W, L is described by a matrix A.

The matrix of a linear transformation $L : V \to W$ is built as follows:

1. Fix a basis $C = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V and a basis $D = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of W.

2. For each v_i in the C-basis, write $L(\mathbf{v}_i)$ as a linear combination $\sum a_{ij}\mathbf{w}_j$ of the D-basis vectors, and

   put the coordinates into a column vector: $[L\mathbf{v}_i]_D = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$

3. Build the matrix A whose columns are $[L\mathbf{v}_1]_D, [L\mathbf{v}_2]_D$, etc. :

$$A = [L]_{DC} = \begin{pmatrix} | & | & \cdots & | \\ L\mathbf{v}_1 & L\mathbf{v}_2 & \cdots & L\mathbf{v}_n \\ | & | & \cdots & | \end{pmatrix}$$

   This is called the *matrix of L with respect to the bases C and D.*

   Then L is then given by multiplication by the matrix A in the following sense:

Theorem 4.2.2 For a vector $\mathbf{x} \in V$ with coordinates $[\mathbf{x}]_C$ in the basis C, the D-coordinates of $L\mathbf{x}$ are given by

$$[L\mathbf{x}]_D = A[\mathbf{x}]_C$$

where $A = [L]_{DC}$ is the above matrix.

Most Useful Case. For linear transformations $L : \mathbb{R}^n \to \mathbb{R}^m$, we can use the standard bases $\{\mathbf{e}_i\}$. The procedure for finding the matrix is then easy:

1. Write $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots$ as a column vectors.

2. Assemble these column vectors into a matrix A.

Then $L(\mathbf{x}) = A\mathbf{x}$, i.e. the transformation is given by matrix multiplication by A.

Example 1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that takes $\mathbf{e}_1 = (1,0)^T$ to $(3,1)^T$ and $\mathbf{e}_2 = (0,1)^T$ to $(1,2)^T$. In column form

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so the matrix of L is $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.   To find where L takes the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, we just multiply by A:

$$L\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$
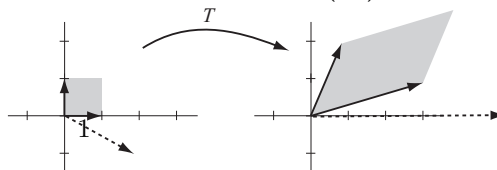
Example 2. What is the matrix of the linear transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - 2y \\ -x + 7y \end{pmatrix}$?

Solution: taking $x = 1$ and $y = 0$, and then $x = 0$ and $y = 1$, we obtain:

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}, \quad \text{so the matrix is} \quad A = \begin{pmatrix} 4 & -2 \\ -1 & 7 \end{pmatrix}.$$

Examples of 6 types of linear transformations $L : \mathbb{R}^2 \to \mathbb{R}^2$.

1. *Dilation* by factor of 3 horizontally, factor of 5 vertically: $D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$.

2. *Rotation* counterclockwise by 90°: $R_{90} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and by angle θ: $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

3. The *reflection* across the x-axis: $R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

4. A *shear* that fixes the first basis vector $\mathbf{e}_1$ and moves $\mathbf{e}_2$ to $\mathbf{e}_1 + \mathbf{e}_2$: $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

5. The *orthogonal projection* onto the x-axis: $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

6. The *embedding* $E : \mathbb{R}^2 \to \mathbb{R}^3$ that takes $\mathbb{R}^2$ into the xy-plane: $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Supplemental Homework Problems

1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that takes $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $3\mathbf{e}_1 + 4\mathbf{e}_2$ and takes $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $-2\mathbf{e}_1 + 5\mathbf{e}_2$. What is the matrix for L with respect to the (standard) basis $\{\mathbf{e}_1, \mathbf{e}_2\}$?

2. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation about the origin by 60°.

   (a) What is $R(\mathbf{e}_1)$? What is $R(\mathbf{e}_2)$?

   (b) What is the matrix of R with respect to the standard basis?

   (c) What is the image $R(\mathbf{v})$ of the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

3. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the yz plane, given by $P(x, y, z) = (0, y, z)$.

   (a) What are $P(\mathbf{e}_1)$, $P(\mathbf{e}_2)$ and $P(\mathbf{e}_3)$?

   (b) What is the matrix of L with respect to the standard basis?

   (c) What is the image $P(\mathbf{v})$ of the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$?

4. (a) Using the formula given in Example 2 above, find the matrix of the rotation $R : \mathbb{R}^2 \to \mathbb{R}^2$ about the origin by 30°.

   (b) This rotation takes $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$ to what vector?

5. Given points $\mathbf{v}_0$ and $\mathbf{v}_1$ in a vector space, the line segment between them is

$$S = \{v_t = (1-t)\mathbf{v}_0 + t\mathbf{v}_1 \mid 0 \le t \le 1\}.$$

Prove that a linear transformation $L : V \to W$ takes each line segment to the line segment between the image points $L(\mathbf{v}_0)$ and $L(\mathbf{v}_1)$.

6. Let $\square$ denote the unit square in $\mathbb{R}^2$ with corners at $(0,0), (1,0), (0,1)$ and $(1,1)$. Using the fact that you proved in Problem 4, sketch the image of $\square$ under the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

7. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the z-axis. What are $P(\mathbf{e}_1), P(\mathbf{e}_2)$ and $P(\mathbf{e}_3)$? What is the matrix for P?

8. The "anti-diagonal" line L^- in $\mathbb{R}^2$ is the graph of $y = -x$, which can also be defined as

$$L^- = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}.$$

Reflection through L^- (obtained by moving along the line segment perpendicular to L^-, going an equal distance to the opposite side of L^-) is a linear transformation, which we denote by R.

(a) What is $R\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? What is $R\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

(b) Find the matrix of this reflection R.

(c) Where does R take the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$?

9. Find a non-zero 2×2 matrix A such that $A\mathbf{v}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for all $\mathbf{v} \in \mathbb{R}^2$. *Hint: Find a vector $\mathbf{w}$ perpendicular to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and define A by $A\mathbf{e}_1 = \mathbf{w}$ and $A\mathbf{e}_2 = r\mathbf{w}$ for your favorite $r \in \mathbb{R}$.*

10. Find the matrices of the following transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$.

(a) The reflection across the xz-plane.

(b) The rotation about the z-axis through an angle θ counterclockwise as viewed from the positive z-axis.

(c) Reflection across the plane $y = z$.

11. Let D be the dilation $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$. Show that D takes the unit circle $x^2 + y^2 = 1$ to an ellipse. Sketch the ellipse. *Hint: What is the image point $\begin{pmatrix} z \\ w \end{pmatrix}$ of $\begin{pmatrix} x \\ y \end{pmatrix}$? What equation do z and w satisfy?*

12. Let $T : P_3 \to P_3$ be the linear transformation that takes a polynomial $p(x) = ax^3 + bx^2 + cx + d$ to $p(x-2)$ (i.e. replace x by $x-2$).

What is the matrix for T with respect to the basis $\{x^3, x^2, x, 1\}$ of P_3? (Be sure to keep the basis elements in this order.)