MTH 234 Quiz 4 Name ____________________
June 16, 2016 (20 points total)

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1. (4 points) Compute the line integral $\int_\gamma x\ ds$ where γ is the triangle with positive orientation connecting the points $(0,0),(1,1)$ and $(1,3)$.

Solution. First come up with parametrizations around the triangle

$$r_1(t) = (t,t) \quad v = \sqrt{2}$$

$$r_2(t) = (1, 2t+1) \quad v = 2$$

$$r_3(t) = (1-t, 3-3t) \quad v = \sqrt{10}$$

Then compute the line integral

$$\int_\gamma = \int_1 + \int_2 + \int_3 = \sqrt{2}\int_0^1 t\ dt + 2\int_0^1\ dt + \sqrt{10}\int_0^1 (1-t)\ dt =$$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2} + 2$$

2. (4 points) Compute the line integral of the vector field $\langle yz, (1+x)z, 1+(1+x)y\rangle$ along the curve $C = \{(t, 2t, 5t^2) : 0 \le t \le 2\}$.

Solution. The given vector field has no curl, and its potential is $\phi = z + yz + xyz$. The fundamental theorem for line integrals then gives
$\int_C \nabla\phi \bullet \vec{dr} = \phi(r(2)) - \phi(r(0)) = 20 + 80 + 160 = 260.$

3. (4 points) Compute the mass of the solid tetrahedron T with vertices $(0,0,0),(2,0,0),(0,1,0)$, and $(0,0,1)$ and with density function $\rho = 2y$.

Hint: It's easier to determine the limits of integration if you draw a picture first.

Solution.
You need to first compute the equation of the plane giving the bounds in the z direction, which requires taking a cross product of the vectors (0,0,1)-(2,0,0) and (0,0,1)-(0,1,0). The equation of the plane is $2z = 2 - x - 2y$ therefore the desired mass integral is

$$\int_0^1 \int_0^{2-2y} \int_0^{\frac{1}{2}2-x-2y} 2y\ dzdxdy =$$

$$\int_0^1 \int_0^{2-2y} y(2-x-2y)\ dxdy = \int_0^1 [2y(2-2y) - \frac{1}{2}y(2-2y)^2 - 2y^2(2-2y)]dy =$$

$$\int_0^1 2y + 2y^3 - 4y^2\ dy =$$

$$1 + \frac{1}{2} - \frac{4}{3} = \frac{1}{6}$$

4. (4 points) The formula for the volume of a right circular cone with radius b and height a is $\frac{1}{3}\pi b^2 a$. Show that this is true by computing a triple integral

$$\iiint_C dV$$

where C is the solid cone lying below the surface $z = a - \frac{a}{b}\sqrt{x^2+y^2}$ and above $z = 0$.

Solution.

$$V = \int_0^{2\pi} \int_0^b \int_0^{a-\frac{a}{b}r} r \; dzdrd\theta = \int_0^{2\pi} \int_{0^b ra - \frac{a}{b}r} r^2 \; drd\theta$$

$$= 2\pi \left(\frac{1}{2}ab^2 - \frac{1}{3}\frac{a}{b}b^3 \right) = \frac{1}{3}\pi ab^2$$

5. (4 points.) Express the volume of the part of the ball $B = \{\rho \leq 8\}$ which lies between the cones $z = r$ and $z = \frac{1}{\sqrt{3}}r$ as a triple integral in spherical coordinates, and evaluate the integral (where $r^2 = x^2 + y^2$ and $\rho^2 = r^2 + z^2$, as usual.)

Solution.

$$V = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_0^8 \rho^2 \sin\phi \; d\rho d\theta d\phi = 2\pi \frac{1}{3} 8^3 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin\phi \; d\phi = \frac{\pi}{3} 8^3 (\sqrt{3} - 1)$$