MTH 234 Quiz 3 *Solutions*
June 9, 2016 (20 points total)

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1. (4 points) Find and **classify** all of the critical points of

$$f(x,y) = xye^{y-x}$$

Given

$$f_{xx} = y(x-2)e^{y-x} \qquad f_{xy} = (1-x)(1+y)e^{y-x} \qquad f_{yy} = x(y+2)e^{y-x}$$

Solution. $\nabla f = \langle (1-x)ye^{y-x}, x(1+y)e^{y-x}\rangle = (0,0)$ at $(x,y) = (0,0)$ or $(x,y) = (1,-1)$. At $(1,-1)$ the hessian is $f_{xx}(1,-1)f_{yy}(1,-1) - f_{xy}^2(1,-1) = e^{-2}e^{-2} - 0 = e^{-4} > 0$ and $f_{xx}(1,-1) > 0$ so $(1,-1)$ is a minimum. At $(0,0)$, the hessian is $0 \cdot 0 - 1^2 = -1 < 0$ so $(0,0)$ is a saddle.

2. (4 points) Find the minimum and maximum value of $f(x,y) = 4 - x^2 - y^2$ subject to $g(x,y) = x^2 + 2y^2 = 1$ using the method of Lagrange multipliers.

Hint: when you solve the equation $\nabla f = \lambda \nabla g$, break the result down into cases:
Case 1: either $x = ?$ or $\lambda = ?$.
Case 2: either $y = ?$ or $\lambda = ?$.
Then analyze these cases to finish the problem.

Solution. The Lagrange multiplier condition gives $-2x = 2\lambda x$ and $-2y = 4\lambda y$. We then have case 1: either $x = 0$ or $\lambda = -1$, and case 2: either $y = 0$ or $\lambda = -1/2$. Note that $(x,y) = (0,0)$ is impossible since $g(0,0) \neq 1$. In the first case, we have $\lambda = -1 \implies y = 0$ so that $x = \pm 1$. In the second case, we have $\lambda = -1/2 \implies x = 0$ so that $y = \pm\frac{1}{\sqrt{2}}$. We then evaluate to find $f(\pm 1, 0) = 3$ and $f(0, \pm 1/\sqrt{2}) = 7/2$. So the minimum of f on $g = 1$ is 3 and the maximum is $7/2$.

3. (4 points) Evaluate

$$\iint_D y \sin x \ \mathrm{d}A$$

where D is the region enclosed by the curves $x = -\pi/2$,$y = 0$, $x = y^2$, and $y = \sqrt{\pi/6}$.

(Hint: you can either do a type I or a type II integral. One method requires fewer steps than the other.)

Solution.

$$\int_0^{\sqrt{\pi/6}} \int_{-\pi/2}^{y^2} y \sin x \ dxdy = -\int_0^{\sqrt{\pi/6}} y\cos(y^2)dy =$$

$$-\frac{1}{2}\int_0^{\pi/6} \cos u \ du = -\frac{1}{4}.$$

4. (4 points) Find the volume of the solid beneath the surface $z = xy$ and above the triangle in the xy-plane with vertices $(0,1)$, $(3,1)$ and $(0,2)$.

Solution. The line connecting $(0,2)$ and $(3,1)$ is $x = 6 - 3y$. We then have

$$V = \int_1^2 \int_0^{6-3y} xy \; dxdy = \frac{1}{2} \int_1^2 y(6-3y)^2 \; dy =$$

$$\frac{9}{2} \int_1^2 y(4 - 4y + y^2) \; dy = \frac{9}{2} \left(2y^2 - \frac{4}{3}y^3 + \frac{1}{4}y^4 \right) \Big|_1^2 = \frac{9}{2} \left(6 - \frac{28}{3} + \frac{15}{4} \right).$$

5. (4 points) Find the volume of the solid beneath the surface $z = x^2 + y^2 + 1$ and above the region bounded by the unit circle $x^2 + y^2 = 1$ in the xy-plane. (Hint: This problem is easier to solve using polar coordinates, but you do not need to.)

Solution.

$$V = \int_0^{2\pi} \int_0^1 (r^2 + 1) r dr d\theta = \int_0^{2\pi} \left(\frac{1}{4}r^4 + \frac{1}{2}r^2 \right) \Big|_0^1 \; d\theta = \int_0^{2\pi} \frac{3}{4} \; d\theta = \frac{3\pi}{2}.$$