Math 2401 Name:________________________

Exam 4

Section B

Number

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:________________________

Problem	Possible	Earned
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. (10 pts) Let C be the curve traced out by $\vec{r}(t) = (\cos t + t\sin t, \sin t - t\cos t)$ with $0 \leq t \leq \sqrt{3}$ and let $f(x, y) = \sqrt{x^2 + y^2}$. Compute

$$\int_C f\, ds.$$

Solution: Note that

$$f(\vec{r}(t)) = \sqrt{(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2} = \sqrt{1 + t^2}.$$

Also, we have that

$$\frac{d\vec{r}}{dt}(t) = (-\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t) = (t\cos t, t\sin t)$$

and so $\left\|\frac{d\vec{r}}{dt}(t)\right\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = |t| = t$ since $0 \leq t \leq \sqrt{3}$. Thus,

$$\begin{aligned}
\int_C f\, ds &= \int_0^{\sqrt{3}} f(\vec{r}(t)) \left\|\frac{d\vec{r}}{dt}(t)\right\| dt \\
&= \int_0^{\sqrt{3}} t\sqrt{1+t^2}\, dt \\
&= \frac{1}{2}\int_1^4 u^{\frac{1}{2}}\, du \\
&= \frac{1}{2} \cdot \frac{2}{3}\, u^{\frac{3}{2}}\Big|_{u=1}^{4} \\
&= \frac{1}{3}\left(4^{\frac{3}{2}} - 1\right) = \frac{7}{3}.
\end{aligned}$$

Grading Metric:
2 points for computing $f(\vec{r}(t))$
1 points for computing $\vec{r}\,'(t)$
2 points for computing ds
3 points for having the correct integral set up
- 1 point for limits
- 1 point for recognizing $t \geq 0$
- 1 point for integrand

2 points for evaluating the integral correctly.

2. (10 pts) Let C be the curve traced out by $\vec{r}\,(t) = (t, t^2)$ with $0 \le t \le b$ and let $\vec{F}\,(x, y) = (xy, x^2 + y)$. Compute

$$\int_C \vec{F} \cdot d\vec{r}.$$

Solution: Note that

$$d\vec{r} = (dx, dy) = (dt, 2t\,dt).$$

So we have

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^b F_1(x(t), y(t))\,dx + F_2(x(t), y(t))\,dy \\
&= \int_0^b [t \cdot t^2 dt + (t^2 + t^2)2t\,dt] \\
&= \int_0^b 5t^3\,dt \\
&= \frac{5}{4}t^4 \Big|_{t=0}^{b} \\
&= \frac{5}{4}b^4.
\end{aligned}$$

Grading Metric:
3 points for computing $\vec{F}(r(t))$
2 points for computing $d\vec{r}$
3 points for having the correct integral set up
2 points for evaluating the integral correctly.

3. (10 pts) Let $a > 0$ and C denote the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,a)$ oriented counter-clockwise. Show that

$$\oint_C \sqrt{1+x^3}\,dx + 2xy\,dy = \frac{a^2}{3}.$$

Solution: If you try to evaluate this directly, you will not be able compute some of the resulting integrals.

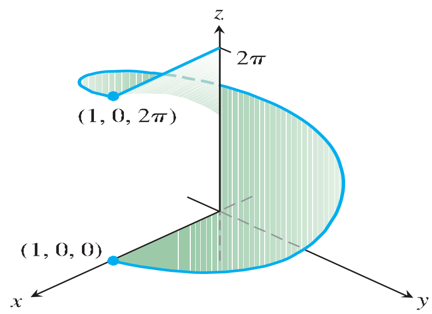
So instead we apply Green's Theorem. Let $\Omega = \{(x,y) : 0 \le x \le 1, 0 \le y \le ax\}$ and note that the boundary of Ω is the curve C. By Green's Theorem, we have

$$\begin{aligned}
\oint_C \sqrt{1+x^3}\,dx + 2xy\,dy &= \iint_\Omega \left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(\sqrt{1+x^3}\right)\right) dA(x,y) \\
&= \iint_\Omega 2y\,dA(x,y) \\
&= \int_{x=0}^1 \left(\int_{y=0}^{ax} 2y\,dy\right) dx \\
&= \int_{x=0}^1 y^2\Big|_0^{ax}\,dx \\
&= a^2 \int_0^1 x^2\,dx = \frac{a^2}{3}x^3\Big|_0^1 = \frac{a^2}{3}.
\end{aligned}$$

Grading Metric:
3 points for determining Ω
3 points for applying Green's Theorem correctly
2 points for setting up the double integral as a correct iterated integral
2 points for evaluating the integral correctly.

Grading Metric used if Green's Theorem is Not Applied
1 point for setting up each integral correctly
1 point for evaluating each integral

4. (10 pts) Let S be the helicoid surface given by $\vec{r}(u,v) = (u\cos v, u\sin v, v)$ with $0 \le v \le 2\pi$ and $0 \le u \le 1$ sketched in the picture below:



Evaluate $\iint_S \sqrt{1+x^2+y^2}\, d\sigma$.

Solution: (a) We have:

$$\begin{aligned} \frac{\partial \vec{r}}{\partial u} &= (\cos v, \sin v, 0) \\ \frac{\partial \vec{r}}{\partial v} &= (-u\sin v, u\cos v, 1). \end{aligned}$$

(b)

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{pmatrix} = (\sin v, -\cos v, u).$$

(c) $d\sigma = \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| \, dA(u,v) = \sqrt{1+u^2}\, dA(u,v)$.

(d) By the above, we have

$$\begin{aligned} \iint_S \sqrt{1+x^2+y^2}\, d\sigma &= \int_{u=0}^{1} \int_{v=0}^{2\pi} \sqrt{1+u^2}\sqrt{1+u^2}\, du\, dv \\ &= 2\pi \int_0^1 (1+u^2)\, du = 2\pi \left(u + \frac{u^3}{3} \right) \Bigg|_{u=0}^{1} = \frac{8\pi}{3}. \end{aligned}$$

Grading Metric:
1 point for computing $\partial_u \vec{r}$ and $\partial_v \vec{r}$
2 point for computing $\partial_u \vec{r} \times \partial_v \vec{r}$
1 point for computing $\|\partial_u \vec{r} \times \partial_v \vec{r}\|$
2 points for computing $d\sigma$
2 points for setting up the surface integral correctly
2 points for evaluating the integral correctly.

5. (10 pts) Let $\vec{F}(x, y, z) = (2x\ln y - yz, \frac{x^2}{y} - xz, -xy)$.

(a) Show that $\vec{F}$ is a conservative vector field;

(b) Find the scalar potential function f such that $\vec{F} = \nabla f$;

(c) Evaluate the line integral of $\vec{F}$ over any path $\vec{r}$ starting at the point $(0, 1, 0)$ and ending at the point $(1, e, -1)$. Namely, compute

$$\int_C \vec{F} \cdot d\vec{r}$$

where C is any path starting at $(0, 1, 0)$ and ending at $(1, e, -1)$.

Solution: (a) We have that $F_1(x, y, z) = 2x\ln y - yz$, $F_2(x, y, z) = \frac{x^2}{y} - xz$, and $F_3(x, y, z) = -xy$ and so

$$\frac{\partial F_1}{\partial y} = \frac{2x}{y} - z = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = -y = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = -x = \frac{\partial F_3}{\partial y}.$$

So $\vec{F}$ is conservative.

(b) We want f so that $\nabla f = \vec{F}$. This implies

$$\frac{\partial f}{\partial x} = F_1(x, y, z) = 2x\ln y - yz.$$

So $f(x, y, z) = x^2 \ln y - xyz + H(y, z)$. But, then

$$\frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial H}{\partial y}(y, z)$$

and since $\frac{\partial f}{\partial y} = F_2(x, y, z) = \frac{x^2}{y} - xz$ we see that $\frac{\partial H}{\partial y}(y, z) = 0$ or $H(y, z) = G(z)$ and so $f(x, y, z) = x^2 \ln y - xyz + G(z)$. However, this gives

$$\frac{\partial f}{\partial z} = -xy + G'(z)$$

and since $\frac{\partial f}{\partial z} = F_3(x, y, z) = -xy$ we see that $G'(z) = 0$ or $G(z) = C$. Thus,

$$f(x, y, z) = x^2 \ln y - xyz + C$$

is the scalar potential we seek.

(c) By the Fundamental Theorem of Line Integrals we have

$$\int_C \vec{F} \cdot d\vec{r} = f(1, e, -1) - f(0, 1, 0) = 1^2 \ln 1 - 1 \cdot e \cdot (-1) + C - (0^2 \ln 1 - 0 \cdot 1 \cdot 0 + C) = 1 + e.$$

Grading Metric:
3 points for showing that the vector field is conservative
4 point for computing f
3 points for using Fundamental Theorem of Line Integrals correctly and getting correct answer