Worksheet 5 – Solutions

① $\vec{r}(t) = \langle t\sin t + \cos t\ ,\ -t\cos t + \sin t \rangle$; $-\sqrt{2} \le t \le 2$

a). $\vec{v}(t) = \langle \sin t + t\cos t - \sin t\ ,\ -\cos t + t\sin t + \cos t \rangle$

$= \langle t\cos t\ ,\ t\sin t \rangle$

b). $|\vec{v}(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t|$

$\vec{T}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \left\langle \dfrac{t\cos t}{|t|}\ ,\ \dfrac{t\sin t}{|t|} \right\rangle$

Be careful with $\sqrt{t^2}$!

$\boxed{\sqrt{t^2} = |t|}$ ← correct

$\sqrt{t^2} = t$ ← incorrect!

Example: $\sqrt{9} = \sqrt{(-3)^2} \neq -3$

$= 3 = |-3|$

c). $L = \displaystyle\int_{-\sqrt{2}}^{2} |\vec{v}(t)|\,dt = \int_{-\sqrt{2}}^{2} |t|\,dt$

$= \displaystyle\int_{-\sqrt{2}}^{0} (-t)\,dt + \int_{0}^{2} t\,dt$

$= -\dfrac{t^2}{2}\Big|_{-\sqrt{2}}^{0} + \dfrac{t^2}{2}\Big|_{0}^{2} = (0+1) + (2-0) = \boxed{3}$

② $\vec{r}(t) = \langle 3\cos t\ ,\ 3\sin t\ ,\ t \rangle$

a). $\vec{v}(t) = \langle -3\sin t\ ,\ 3\cos t\ ,\ 1 \rangle$

b). $|\vec{v}(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$

$\vec{T}(t) = \dfrac{1}{\sqrt{10}} \langle -3\sin t\ ,\ 3\cos t\ ,\ 1 \rangle$

c). $s(t) = \displaystyle\int_0^t |\vec{v}(\tau)|\,d\tau = \int_0^t \sqrt{10}\,d\tau = \sqrt{10}\,\tau\Big|_0^t = \boxed{\sqrt{10}\,t}$

③ $\dfrac{d\vec{r}}{dt} = \left\langle 6\sqrt{t+1}\ ,\ 4e^{-t}\ ,\ \dfrac{1}{t+1} \right\rangle$

$\vec{r}(t) = \left\langle 6\cdot\dfrac{2}{3}(t+1)^{3/2} + C_1\ ,\ -4e^{-t} + C_2\ ,\ \ln(t+1) + C_3 \right\rangle$

$\vec{r}(0) = \langle 4 + C_1\ ,\ -4 + C_2\ ,\ C_3 \rangle$

$\vec{r}(0) = \langle 0\ ,\ 0\ ,\ 1 \rangle$

$\Rightarrow C_1 = -4\ ;\ C_2 = 4\ ;\ C_3 = 1$

$$\boxed{\vec{r}(t) = \left\langle 4(t+1)^{3/2} - 4\ ,\ -4e^{-t} + 4\ ,\ \ln(t+1) + 1 \right\rangle}$$

④ $\vec{r}(t) = \left(\ln\frac{t}{6}\right)\vec{i} + \left(\frac{t-6}{t+7}\right)\vec{j} + \left(t\ln\frac{t}{6}\right)\vec{k}$; $t=6$

$$\vec{v}(t) = \left\langle \frac{1}{t/6}\cdot\frac{1}{6}\,,\ \frac{(t+7)-(t-6)}{(t+7)^2}\,,\ \ln\frac{t}{6} + t\cdot\frac{1}{t/6}\,\frac{1}{6} \right\rangle$$

$$= \left\langle \frac{1}{t}\,,\ \frac{13}{(t+7)^2}\,,\ \ln\left(\frac{t}{6}\right)+1 \right\rangle$$

$\vec{v}(6) = \left\langle \frac{1}{6}, \frac{1}{13}, 1 \right\rangle$ (vector parallel to the line)

Point: $\vec{r}(6) = \langle 0, 0, 0 \rangle$ => $(0,0,0)$ (point on the line)

Equations for the line:

$$\begin{cases} x = \frac{1}{6}\,t \\ y = \frac{1}{13}\,t \\ z = t \end{cases}$$