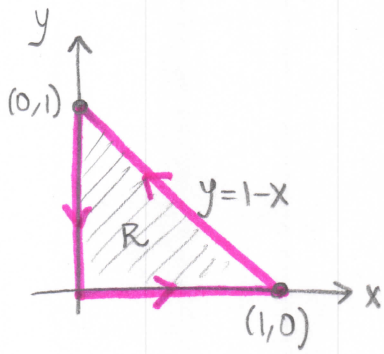
Worksheet 16

1 $\oint_C x^4\,dx + xy\,dy$



Green's Theorem:

$$\oint_C x^4\,dx + xy\,dy = \oint_C M\,dx + N\,dy \qquad \begin{array}{l} M = x^4 \\ N = xy \end{array}$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (y - 0)\,dA$$

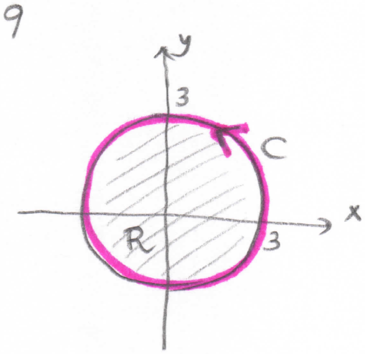
$$= \int_0^1 \int_0^{1-x} y\,dy\,dx$$

$$= \int_0^1 \frac{y^2}{2}\Big|_{y=0}^{y=1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= -\frac{1}{6}(1-x)^3\Big|_0^1 = \boxed{\frac{1}{6}}$$

2 $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$; $C:\ x^2+y^2=9$

$$= \oint_C M\,dx + N\,dy \qquad \begin{array}{l} M = 3y - e^{\sin x} \\ N = 7x + \sqrt{y^4+1} \end{array}$$



$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (7 - 3)\,dA$$

$$= \iint_R 4\,dA$$

$$= 4\,\text{Area}(R)$$

$$= 4\,(\pi \cdot 3^2)$$

$$= \boxed{36\pi}$$

3 Area enclosed by an ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

$C: \vec{r}(t) = \langle a\cos t, b\sin t\rangle, \ 0 \le t \le 2\pi$

$$x = a\cos t \ ; \ dx = -a\sin t\, dt$$
$$y = b\sin t \ ; \ dy = b\cos t\, dt$$

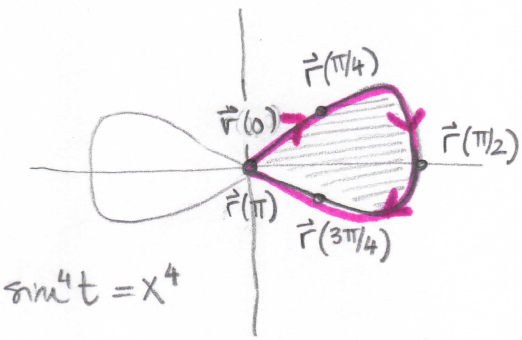
$$\text{Area} = \frac{1}{2}\oint x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t) - (b\sin t)(-a\sin t)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,(\cos^2 t + \sin^2 t)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, dt = \frac{1}{2}abt\Big|_0^{2\pi} = \boxed{\pi ab}$$

4 Lemniscate: $x^4 = x^2 - y^2$



(a). $\vec{r}(t) = \langle \sin t, \sin t\cos t\rangle; \ 0 \le t \le \pi.$

$x = \sin t \ ; \ y = \sin t\cos t$

$$x^2 - y^2 = \sin^2 t - \sin^2 t\cos^2 t = \sin^2 t\,\underbrace{(1 - \cos^2 t)}_{\sin^2 t} = \sin^4 t = x^4$$

The restriction $0 \le t \le \pi$ means $\sin t \ge 0$
so $x = \sin t \ge 0$ (the right lobe).

(b). $\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{\pi} \Big((\sin t)(2\cos^2 t - 1) - \sin t\cos t \cdot \cos t\Big)\, dt$

$x = \sin t \ ; \ dx = \cos t\, dt$

$y = \sin t\cos t = \frac{1}{2}\sin(2t) \ ; \ dy = \cos(2t)\,dt = (\cos^2 t - \sin^2 t)\,dt = (2\cos^2 t - 1)\,dt$

$$= \frac{1}{2}\int_0^{\pi} \left(2\sin t\cos^2 t - \sin t - \sin t\cos^2 t\right) dt$$

$$= \frac{1}{2}\int_0^{\pi} (\sin t\cos^2 t - \sin t)\, dt$$

$$= \frac{1}{2}\left(-\frac{1}{3}\cos^3 t + \cos t\right)\Big|_0^{\pi}$$

$$= \frac{1}{2}\left(\frac{1}{3} - 1 + \frac{1}{3} - 1\right) = \frac{1}{2}\left(\frac{2}{3} - 2\right) = -\frac{2}{3} \quad ?$$

What happened? The parametrization has the curve negatively oriented:

$\vec{r}(0) = \langle 0,0\rangle \ ; \ \vec{r}\left(\frac{\pi}{4}\right) = \left\langle \frac{\sqrt{2}}{2}, \frac{1}{2}\right\rangle \ ; \ \vec{r}\left(\frac{\pi}{2}\right) = \langle 1,0\rangle ; \ \vec{r}\left(\frac{3\pi}{4}\right) = \left\langle \frac{\sqrt{2}}{2}, -\frac{1}{2}\right\rangle ; \ \vec{r}(\pi) = \langle 0,0\rangle$

So really:

$$\text{Area} = -\left(\frac{1}{2}\oint_C x\,dy - y\,dx\right) \Rightarrow \text{Area} = \boxed{2/3}$$