Name: Solutions

April 30^{th}, 2015.
Math 2401; Sections K1, K2, K3.
Georgia Institute of Technology
FINAL EXAM

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Total	140	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Angle θ $(0 \le \theta \le \pi)$ between vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Vector Projection of $\mathbf{u}$ onto $\mathbf{v} \neq 0$:

$$\mathrm{Proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = |\mathbf{u}\cos\theta|\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Distance from a point S to a line L going through P and parallel to $\mathbf{v}$:

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$$

Length of a smooth curve C: $\mathbf{r}(t)$, traced exactly once as $a \le t \le b$:

$$L = \int_a^b |\mathbf{v}(t)|\,dt.$$

TNB Frame:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|};\quad \mathbf{N} = \frac{d\mathbf{T}/ds}{\kappa} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|};\quad \mathbf{B} = \mathbf{T}\times\mathbf{N}.$$

Curvature:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

Tangential and Normal Components of Acceleration:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N};$$

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}(t)|;$$

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}(t)|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}.$$

Torsion:

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}.$$

Directional Derivative of f at P_0 in the direction of the unit vector $\mathbf{u}$:

$$(D_{\mathbf{u}}f)_{P_0} = (\nabla f)_{P_0}\cdot\mathbf{u}.$$

Spherical Coordinates: (ρ, ϕ, θ):

$$0 \le \phi \le \pi;\quad 0 \le \theta \le 2\pi;$$

$$x = \rho\sin\phi\cos\theta;\quad y = \rho\sin\phi\sin\theta;\quad z = \rho\cos\phi;$$

$$\text{Jacobian: } dV \mapsto \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Green's Theorem in the Plane:

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dA;$$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

Area with Green's Theorem:

$$\mathrm{Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Surface Differential on Parametric Surface S : $\mathbf{r}(u,v)$; $(u,v) \in R$:

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\,d(u,v)$$

Unit Normal Field on Parametric Surface S : $\mathbf{r}(u,v)$; $(u,v) \in R$:

$$\mathbf{n} = \pm\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$$

Surface Differential on Implicitly Defined (Level Surface) $f(x,y,z) = c$, over shadow region R in a coordinate plane:

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA,$$

where $\mathbf{p}$ is one of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.
Unit Normal Field on Implicitly Defined (Level Surface) $f(x,y,z) = c$, over shadow region R in a coordinate plane:

$$\mathbf{n} = \pm\frac{\nabla f}{|\nabla f|}.$$

Parametrized Sphere of radius R, centered at the origin: $0 \le \phi \le \pi$; $0 \le \theta \le 2\pi$;

$$\mathbf{r}(\phi,\theta) = R\langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle;$$

$$\mathbf{r}_\phi\times\mathbf{r}_\theta = R^2\langle\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle;$$

$$|\mathbf{r}_\phi\times\mathbf{r}_\theta| = R^2\sin\phi.$$

Stokes' Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma,$$

(with the appropriate assumptions on C, S and $\mathbf{F}$.)

Divergence Theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV,$$

(with the appropriate assumptions on S, D and $\mathbf{F}$.)

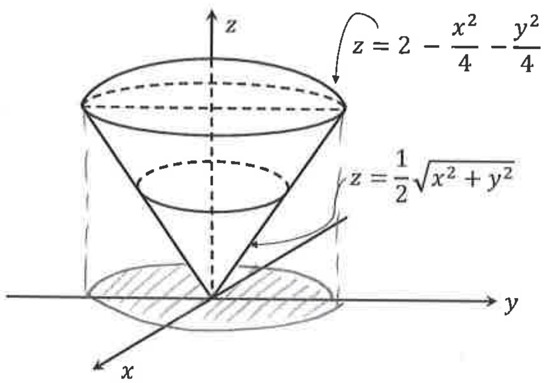
1. [10 points] Set up a triple integral in *cylindrical coordinates* that gives the volume of the "ice cream cone," the solid bounded by the cone

$$z = \frac{1}{2}\sqrt{x^2+y^2}$$

and the paraboloid

$$z = 2 - \frac{x^2}{4} - \frac{y^2}{4}.$$

You do not need to compute the integral, just set it up!



$z = 2 - \frac{r^2}{4}$ (2 pts.)

$z = \frac{1}{2}r$ (2 pts.)

Find radius of circle of intersection:

$2 - \frac{r^2}{4} = \frac{1}{2}r$ (1 pt.)

$8 - r^2 = 2r$

$r^2 + 2r - 8 = 0$

$(r+4)(r-2) = 0$

$\boxed{r=2}$ (1 pt.)

$$\boxed{\underbrace{\int_0^{2\pi}}_{\text{(1 pt.)}} \underbrace{\int_0^{2}}_{\text{(1 pt.)}} \underbrace{\int_{\frac{1}{2}r}^{2-\frac{r^2}{4}}}_{\text{(1 pt.)}} \underbrace{r\,dz\,dr\,d\theta}_{\text{(1 pt.)}}}$$

2. [10 points] Consider the curve:

$$\vec{r}(t) = (t\sin t + \cos t)\,\vec{i} + (-t\cos t + \sin t)\,\vec{j}; \quad -\sqrt{2} \le t \le 0.$$

a). Find the velocity $\vec{v}(t)$.

$$\vec{v}(t) = \langle \sin t + t\cos t - \sin t,\ -\cos t + t\sin t + \cos t \rangle \quad \text{(2 pts.)}$$

$$= \langle t\cos t,\ t\sin t \rangle \quad \text{(1 pt.)}$$

b). Find the unit tangent vector $\vec{T}(t)$.

(1 pt.) $|\vec{v}(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$

$= \sqrt{t^2(\cos^2 t + \sin^2 t)}$

(1 pt.) $= \sqrt{t^2}$

$= |t|$

(1 pt.) (1/2 pt.) $= -t$ because $t \leq 0$

minus sign t

$$\boxed{\vec{T}(t) = \langle -\cos t,\ -\sin t \rangle} \quad \text{(1/2 pt.)}$$

c). Find the length of the curve.

$$L = \int_{-\sqrt{2}}^{0} |\vec{v}(t)|\,dt \quad \text{(1 pt.)}$$

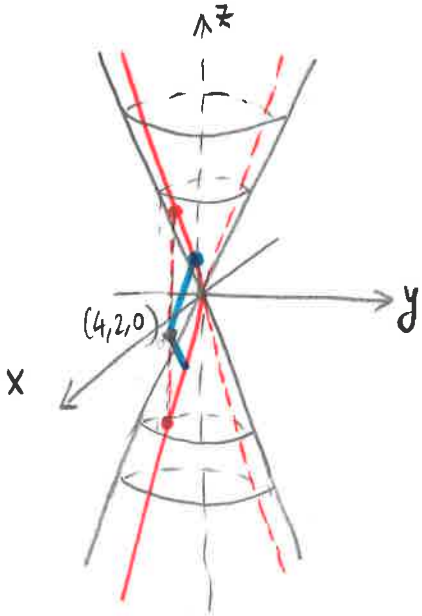
$$= \int_{-\sqrt{2}}^{0} (-t)\,dt$$

$$= -\frac{t^2}{2}\Big|_{-\sqrt{2}}^{0} \quad \text{(1 pt.)}$$

$$= 0 - (-1)$$

$$= \boxed{1} \quad \text{(1 pt.)}$$

3. [10 points] Find the points on the cone $x^2 + y^2 = z^2$ that are closest to the point $(4, 2, 0)$.



Solution 1 : Lagrange Multipliers :

Minimize : distance b/w $(4,2,0)$ and (x,y,z)

subject to constraint : $x^2+y^2=z^2$

$$f(x,y,z) = (x-4)^2+(y-2)^2+z^2$$

$$g(x,y,z) = x^2+y^2-z^2 = 0$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \qquad \begin{aligned} \nabla f &= \langle 2(x-4),\ 2(y-2),\ 2z \rangle \\ \nabla g &= \langle 2x,\ 2y,\ -2z \rangle \end{aligned}$$

$$\begin{cases} 2(x-4) = \lambda \cdot 2x \\ 2(y-2) = \lambda \cdot 2y \\ 2z = \lambda \cdot (-2z) \\ x^2+y^2 = z^2 \end{cases} \qquad \begin{cases} x-4 = \lambda x \\ y-2 = \lambda y \\ z = -\lambda z \longrightarrow z(1+\lambda) = 0 \Rightarrow z=0 \text{ or } \lambda = -1 \\ x^2+y^2 = z^2 \end{cases}$$

$z=0 \Rightarrow x^2+y^2=0 \Rightarrow x=y=0$

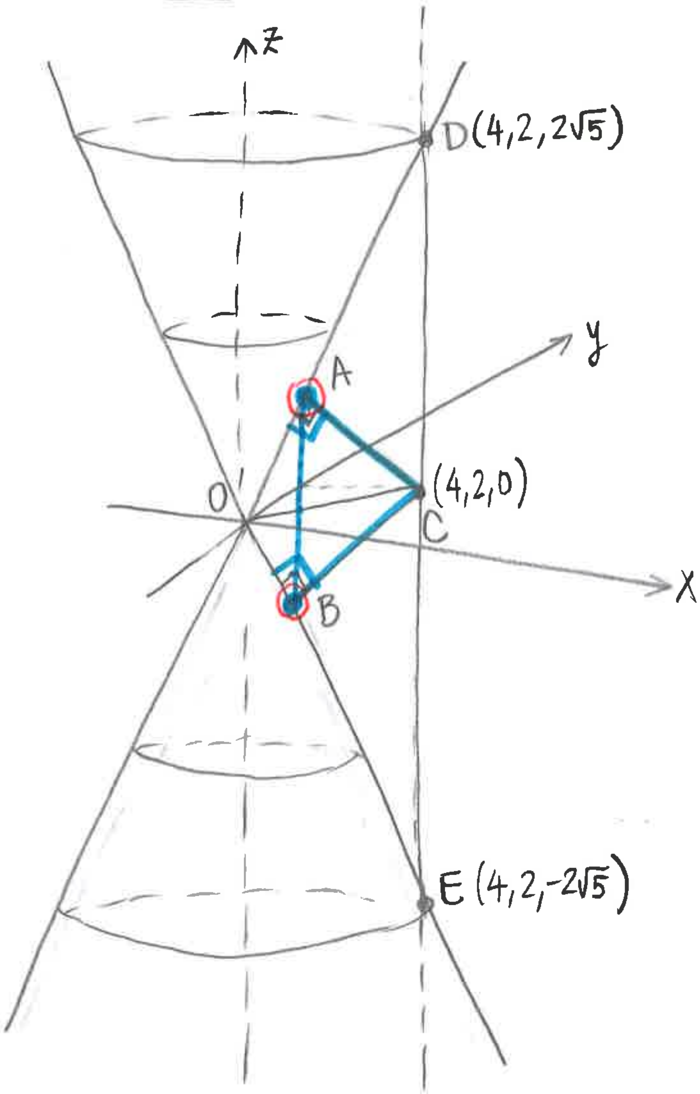
$\Rightarrow$ Eqn. 1 becomes $-4=0$ ↯

$$\Rightarrow \lambda = -1 \Rightarrow \begin{cases} x-4 = -x & \Rightarrow x = 2 \\ y-2 = -y & \Rightarrow y = 1 \\ x^2+y^2 = z^2 & \Rightarrow z^2 = 5 \Rightarrow z = \pm\sqrt{5} \end{cases}$$

Answer: $\boxed{(2,1,\sqrt{5}) \text{ and } (2,1,-\sqrt{5})}$

(2pts.) - correct setup (minimization of distance, constraint)
(1pt.) - correct f, g
(1pt.) - gradients
(1pt.) - Lagrange system setup
(4pts.) - solving system
(1pt.) - final answer

Solution 2 : Geometric Reasoning



Line L, vertical through $(4,2,0)$:

$x=4;\ y=2;\ z=t$

Intersection of L with cone:

$z^2 = 4^2+2^2 = 20 \Rightarrow z = \pm 2\sqrt{5}$

$\Rightarrow D(4,2,2\sqrt{5});\ E(4,2,-2\sqrt{5})$

$\Rightarrow$ A is on the line determined by O & D:

$\langle 4t, 2t, 2\sqrt{5}t \rangle$

$\Rightarrow A(4t, 2t, 2\sqrt{5}t)$ for some t

$\measuredangle OAC$ is a right angle

$\Rightarrow \vec{AO} \cdot \vec{AC} = 0$

$\vec{AO} = \langle 4t, 2t, 2\sqrt{5}t \rangle$

$\vec{AC} = \langle 4-4t, 2-2t, -2\sqrt{5}t \rangle$

$$\vec{AO} \cdot \vec{AC} = 16t - 16t^2 + 4t - 4t^2 - 20t^2$$
$$= 20t - 40t^2$$
$$= 20(t-2t^2)$$
$$= 20t(1-2t) \Rightarrow t = 1/2$$

$\Rightarrow \boxed{A(2,1,\sqrt{5})}$

$\Rightarrow \boxed{B(2,1,-\sqrt{5})}$ (mirror image through xy-plane).

4. [10 points] Given that for a curve $\mathbf{r}(t)$:

$$\frac{d\mathbf{r}}{dt} = 3\sqrt{t+1}\,\mathbf{i} + 4e^{-t}\,\mathbf{j} + \frac{1}{t+1}\,\mathbf{k},$$

and that:

$$\mathbf{r}(0) = \langle 1, 0, 2 \rangle\,,$$

find $\mathbf{r}(t)$.

$$\int 3\sqrt{t+1}\,dt = 3\cdot\frac{2}{3}(t+1)^{3/2} + C_1 = 2(t+1)^{3/2} + C_1$$ (2pts.)

$$\int 4e^{-t}\,dt = -4e^{-t} + C_2$$ (2pts.)

$$\int \frac{1}{t+1}\,dt = \ln(t+1) + C_3$$ (2pts.)

$$\vec{r}(t) = \left\langle 2(t+1)^{3/2} + C_1,\ -4e^{-t} + C_2,\ \ln(t+1) + C_3 \right\rangle$$

(4/3pts.) (4/3pts.) (4/3 pts.)

$$\vec{r}(0) = \langle 2 + C_1,\ -4 + C_2,\ C_3 \rangle = \langle 1,\ 0,\ 2 \rangle \Rightarrow C_1 = -1 \quad C_2 = 4 \quad C_3 = 2$$

$$\boxed{\vec{r}(t) = \left\langle 2(t+1)^{3/2} - 1,\ -4e^{-t} + 4,\ \ln(t+1) + 2 \right\rangle}$$

5. [10 points] Find all the critical points of $f(x,y) = xy^2 - x^2 - 2y^2$ and classify each one as either a local minimum, a local maximum, or a saddle point.

(1pt.) $f_x = y^2 - 2x$ (½pt.) $f_{xx} = -2$ $f_{xy} = 2y$ (½pt.)

(1pt.) $f_y = 2xy - 4y$ (½pt.) $f_{yy} = 2x - 4$

(½pt.) $\begin{cases} y^2 - 2x = 0 \\ 2xy - 4y = 0 \end{cases} \Rightarrow 2y(x-2) = 0 \Rightarrow y = 0$ or $x = 2$

$x = 0$ $\quad y^2 = 4 \Rightarrow y = \pm 2$

Critical Points: $(0,0)$; $(2,-2)$; $(2,2)$. (3pts.)

$\Delta = f_{xx}f_{yy} - f_{xy}^2 = -2(2x-4) - 4y^2$

$\Delta(0,0) = 8 > 0$; $f_{xx}(0,0) = -2 < 0 \Rightarrow (0,0)$ is a local max (1pt.)

$\Delta(2,2) = -4 \cdot 4 < 0 \Rightarrow (2,2)$ is a saddle point (1pt.)

$\Delta(2,-2) = -4 \cdot 4 < 0 \Rightarrow (2,-2)$ is a saddle point (1pt.)

6. [10 points] Recall that the angle between two planes is defined to be the angle between their normal vectors. Consider the planes:

$$P_1 : x + y + z = -1;$$

$$P_2 : x + 2y + 3z = -4.$$

a). [3 points] Find the angle between the two planes above (give an exact answer).

(1pt.) $\begin{cases} \vec{n}_1 = \langle 1,1,1 \rangle \\ \vec{n}_2 = \langle 1,2,3 \rangle \end{cases}$

(1pt.) $\begin{cases} |\vec{n}_1| = \sqrt{3} \\ |\vec{n}_2| = \sqrt{14} \end{cases}$

(½pt.) (½pt.)

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{1+2+3}{\sqrt{3}\cdot\sqrt{14}} = \frac{6}{\sqrt{6}\cdot\sqrt{7}} = \frac{\sqrt{6}}{\sqrt{7}}$$

$$\boxed{\theta = \cos^{-1}\left(\sqrt{\tfrac{6}{7}}\right)}$$

b). [7 points] Find parametric equations for the line of intersection of the two planes above.

Vector parallel to the line of intersection: $\vec{n}_1 \times \vec{n}_2$ (1pt.)

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 1, -2, 1 \rangle$$ (2pts.)

Point on line of intersection: $z = 0 \Rightarrow \begin{cases} x + y = -1 \\ x + 2y = -4 \end{cases}$ (3pts.)

$\ominus \quad -y = 3 \Rightarrow y = -3 \Rightarrow x = 2$

Parametric Eqns. of line:

$$\boxed{\begin{cases} x = 2 + t \\ y = -3 - 2t \\ z = t \end{cases}}$$ (1pt.)

7. [10 points] Let $f(x,y)$ have continuous first order partial derivatives. Consider the points:

$$A(1,2);\quad B(2,2);\quad C(1,3);\quad D(5,6).$$

Suppose that:

- the directional derivative of f at the point A in the direction of $\overrightarrow{AB}$ is equal to 2.
- the directional derivative of f at the point A in the direction of $\overrightarrow{AC}$ is equal to 4.

Use this information to find the directional derivative of f at the point A in the direction of $\overrightarrow{AD}$.

3pts. $\overrightarrow{AB} = \langle 1, 0\rangle$ (already unit vector) $\quad D_{\overrightarrow{AB}} f(A) = \nabla f(A)\cdot \overrightarrow{AB} = 2 \quad \nabla f(A) = \langle a, b\rangle$

$\Rightarrow \langle a, b\rangle \cdot \langle 1, 0\rangle = 2 \Rightarrow a = 2$

3pts. $\overrightarrow{AC} = \langle 0, 1\rangle$ (also unit vector) $\quad D_{\overrightarrow{AC}} f(A) = \nabla f(A)\cdot \overrightarrow{AC} = 4$

$\Rightarrow \langle 2, b\rangle \cdot \langle 0, 1\rangle = 4 \Rightarrow b = 4$

(1pt.) $\nabla f(A) = \langle 2, 4\rangle$

$1\frac{1}{2}$pts. $\overrightarrow{AD} = \langle 4, 4\rangle$

$|\overrightarrow{AD}| = 4\sqrt{2}$

$\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

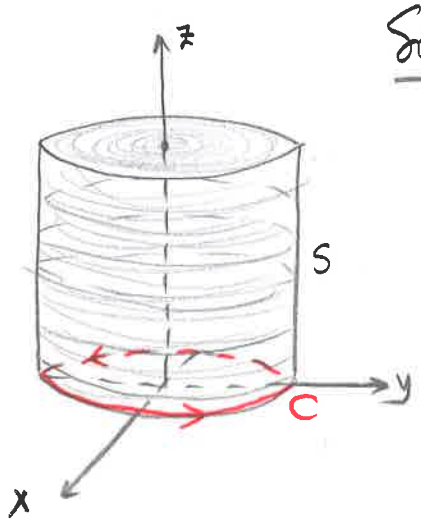
1pt. $D_{\vec{u}} f(A) = \nabla f(A)\cdot \vec{u}$

$= \langle 2, 4\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \underline{\underline{3\sqrt{2}}}$ ($\frac{1}{2}$ pt.)

8. [10 points] Let S be the surface consisting of the cylinder $x^2+y^2=4$, $0 \le z \le 10$, together with its "top," $x^2+y^2 \le 4$, $z=10$. Let:

$$\mathbf{F}(x,y,z) = -2y\mathbf{i} + 2x\mathbf{j} + 2x^2\mathbf{k}.$$

Find the *outward flux* of the curl $\nabla \times \mathbf{F}$ through S.



Solution 1: Stokes' Theorem:

(1pt.) $$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

(1pt.) $C: x^2+y^2=4;\ z=0$

(1pt.) $C: \vec{r}(t) = \langle 2\cos t, 2\sin t, 0 \rangle$

(½ pt.) $0 \le t \le 2\pi$

(2pts.) $\vec{F}(\vec{r}(t)) = \langle -4\sin t, 4\cos t, 4\cos^2 t \rangle$

(2pts.) $\frac{d\vec{r}}{dt} = \langle -2\sin t, 2\cos t, 0 \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$ (1 pt.)

$$= \int_0^{2\pi} (8\sin^2 t + 8\cos^2 t)\, dt$$ (1pt.)

$$= \int_0^{2\pi} 8\, dt$$

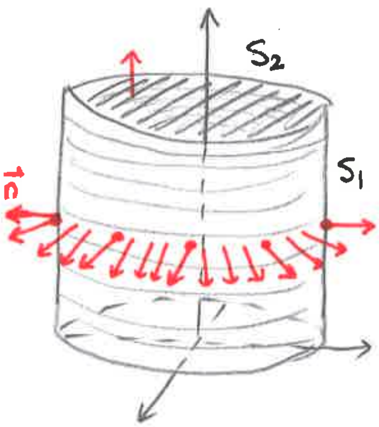
$$= \boxed{16\pi}$$ (½ pt.)

Solution 2 : Direct computation

(2pts.) Curl of $\vec{F}$:

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ -2y & 2x & 2x^2 \end{vmatrix} = \langle 0, -4x, 4\rangle$$

(4pts.) Outward flux through cylinder $x^2+y^2=4$, $0\le z\le 10.$



Parametric Surface : $\vec{r}(\theta,z)=\langle 2\cos\theta, 2\sin\theta, z\rangle$

$0\le\theta\le 2\pi$; $0\le z\le 10$

$$\vec{r}_\theta = \langle -2\sin\theta, 2\cos\theta, 0\rangle$$
$$\vec{r}_z = \langle 0, 0, 1\rangle$$
$$\vec{r}_\theta\times\vec{r}_z = \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

$$\iint_{S_1}(\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_0^{10}\int_0^{2\pi} -4(2\cos\theta)(2\sin\theta)\,d\theta\,dz$$
$$= 10\cdot 8\frac{\cos^2\theta}{2}\Big|_0^{2\pi} = \boxed{0}$$

(4pts.) Outward flux through top disk : $x^2+y^2\le 4$; $z=10$

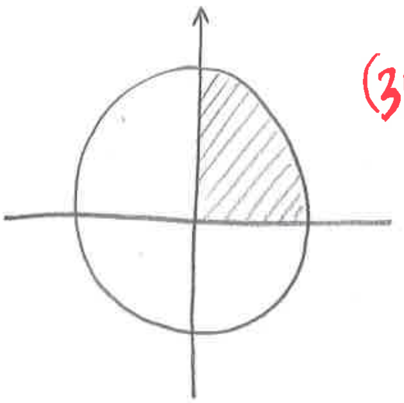
$\vec{n}=\vec{k}$

$$\iint_{S_2}(\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \iint_{S_2}4\,d\sigma = 4(\text{area}(S_2)) = 4\cdot 4\pi = \boxed{16\pi}$$

Total flux : $\iint_S(\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \iint_{S_1}(\nabla\times\vec{F})\cdot\vec{n}\,d\sigma + \iint_{S_2}(\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = 16\pi$

9. [10 points] Compute the integral:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\,dx\,dy.$$



(3pts.)

$0 \le x \le \sqrt{1-y^2}$; $0 \le y \le 1$

$x = \sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2+y^2 = 1$

Switch to polar:

$$\underbrace{\int_0^{\pi/2}}_{1\text{pt.}} \underbrace{\int_0^1}_{1\text{pt.}} \underbrace{\cos(r^2)}_{\frac{1}{2}\text{pt.}} \underbrace{r\,dr\,d\theta}_{\frac{1}{2}\text{pt.}} = \frac{\pi}{2} \cdot \frac{1}{2} \sin(r^2) \Big|_{r=0}^{r=1} \quad \text{(1pt.)}$$

$$= \boxed{\frac{\pi}{4} \sin(1)} \quad \text{(1pt.)}$$

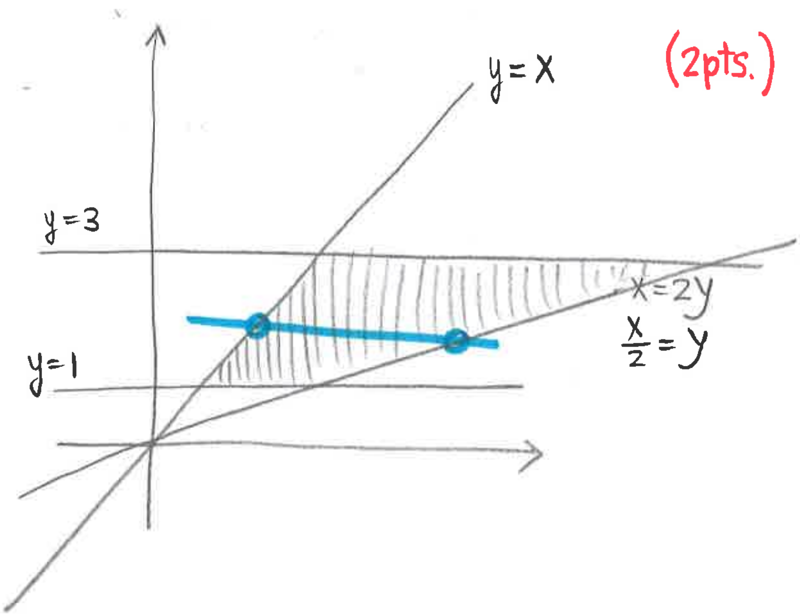
10. [10 points] Compute the integral:

$$\iint_R \frac{\sin y}{y}\, dA,$$

where R is the region in the plane given by:

$$R : 1 \le y \le 3;\ \ y \le x \le 2y,$$

and sketch the region of integration.



Horizontal Cross-Sections : (1pt.)

(Cannot integrate $\frac{\sin y}{y}\,dy$)

(1pt.) (1pt.) (1pt.)

$$\int_1^3 \int_y^{2y} \frac{\sin y}{y}\, dx\, dy$$

$$= \int_1^3 \frac{\sin y}{y} \cdot x \Big|_{x=y}^{x=2y} dy \quad \text{(1pt.)}$$

$$= \int_1^3 \frac{\sin y}{\cancel{y}} \cdot \cancel{y}\, dy \quad \text{(1pt.)}$$

$$= \int_1^3 \sin y\, dy$$

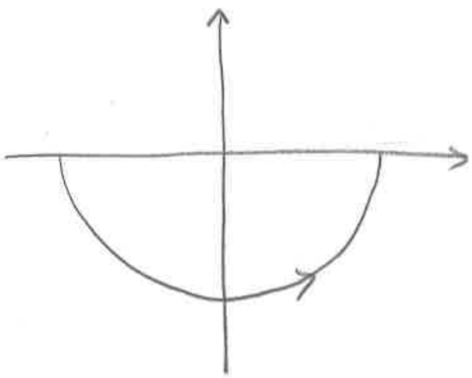
$$= -\cos y \Big|_1^3 \quad \text{(1pt.)}$$

$$= \boxed{\cos(1) - \cos(3)} \quad \text{(1pt.)}$$

11. [10 points] Compute the line integral

$$\int_C (2 + x^2 y)\, ds,$$

where C is the lower half of the unit circle $x^2 + y^2 = 1$ (going from $(-1, 0)$ to $(1, 0)$ along the unit circle, below the x-axis).



Parametrize curve:

$\vec{r}(t) = \langle \cos t, \sin t \rangle$ (1 pt.)

$\pi \leq t \leq 2\pi$ (1 pt.)

$\vec{v}(t) = \langle -\sin t, \cos t \rangle$ (1 pt.)

$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$ (1 pt.)

$$\int_C (2 + x^2 y)\, ds = \int_{\pi}^{2\pi} \left(2 + \cos^2 t \sin t\right) \cdot 1\, dt$$ (3 pts.)

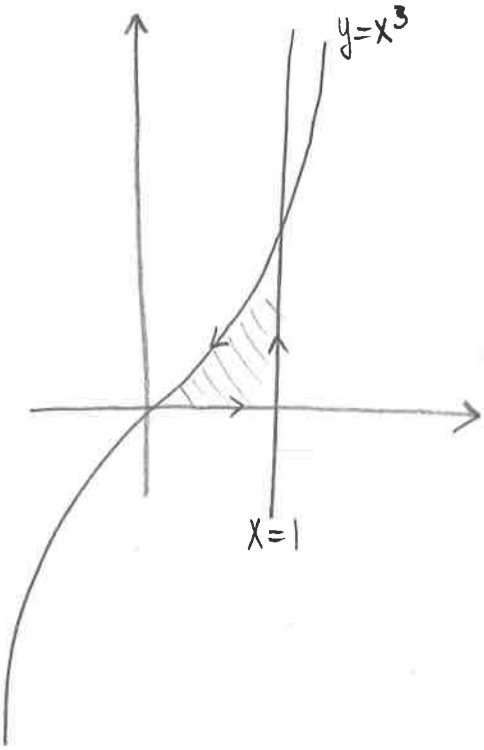
$$= \left(2t - \frac{\cos^3 t}{3}\right)\Bigg|_{\pi}^{2\pi}$$ (2 pts.)

$$= 2\pi - \frac{1}{3} - \frac{1}{3} = \boxed{2\pi - \frac{2}{3}}$$ (1 pt.)

12. [10 points] Find the work done by the field

$$\mathbf{F}(x, y) = 5xy^3\mathbf{i} + 9x^2y^2\mathbf{j}$$

in moving a particle once counterclockwise around the curve C: the boundary of the region enclosed by the x-axis, the line $x = 1$ and the curve $y = x^3$ in the first quadrant.



$\vec{F} = \langle M, N \rangle$

$M(x,y) = 5xy^3$

$N(x,y) = 9x^2y^2$ (1pt.)

Green's Theorem: (work = circulation) (1pt.)

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (18xy^2 - 15xy^2)\, dA \quad \text{(2pts.)}$$

$$= \iint_R 3xy^2\, dA \quad \text{(1pt.)}$$

$$= \int_0^1 \int_{\sqrt[3]{y}}^1 3xy^2\, dx\, dy$$

$$= \int_0^1 3y^2 \frac{x^2}{2}\Big|_{x=\sqrt[3]{y}}^{x=1} dy$$

$$= \int_0^1 \left(3\frac{y^2}{2} - \frac{3}{2} y^2 y^{2/3}\right) dy$$

$$= \frac{3}{2}\left(\frac{y^3}{3} - \frac{3}{11} y^{11/3}\right)\Big|_0^1$$

$$= \frac{3}{2}\cdot\left(\frac{1}{3} - \frac{3}{11}\right) = \frac{3}{2}\cdot\frac{2}{33}$$

$$= \frac{1}{11}$$

or

$$= \int_0^1 \int_0^{x^3} 3xy^2\, dy\, dx \quad \text{(1pt.)}$$

$$= \int_0^1 xy^3 \Big|_{y=0}^{y=x^3} dx \quad \text{(1 pt.)}$$

$$= \int_0^1 x^{10}\, dx \quad \text{(1pt.)}$$

$$= \frac{x^{11}}{11}\Big|_0^1 \quad \text{(1pt.)}$$

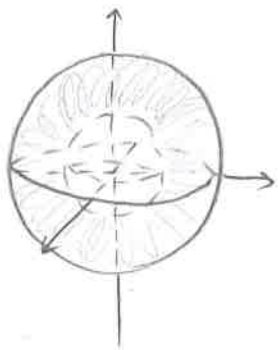
$$= \frac{1}{11} \quad \text{(1pt.)}$$

13. [10 points] Find the outward flux of the field

$$\mathbf{F}(x,y,z) = (2x^3+9xy^2)\,\mathbf{i} + (-y^3+\pi e^y\sin(z))\,\mathbf{j} + (2z^3+\pi e^y\cos(z))\,\mathbf{k},$$

through the boundary of the region D:

$$D:\ 1\le x^2+y^2+z^2\le 2.$$



Divergence Theorem:

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \nabla\cdot\vec{F}\,dV$$ (1pt.)

$$\nabla\cdot\vec{F} = (6x^2+9y^2) + (-3y^2+\cancel{\pi e^y\sin(z)}) + (6z^2-\cancel{\pi e^y\sin(z)})$$ (3pts.)

$$= 6x^2+6y^2+6z^2$$ (1pt.)

$$\iiint_D \nabla\cdot\vec{F}\,dV = \overbrace{\int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}}}^{\text{(1pt.)}} \overbrace{(6\rho^2)}^{\text{(1½pt.)}} \overbrace{(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta}^{\text{(½pt.)}}$$

$$= (2\pi)\,(-\cos\phi)\Big|_0^{\pi}\left(\frac{6\rho^5}{5}\right)\Big|_1^{\sqrt{2}}$$ (1pt.)(½pt.)

$$= (2\pi)(1+1)\left(\frac{6\cdot 4\sqrt{2}}{5}-\frac{6}{5}\right)$$

$$= \boxed{4\pi\cdot\frac{6(4\sqrt{2}-1)}{5}}$$ (½ pt.)

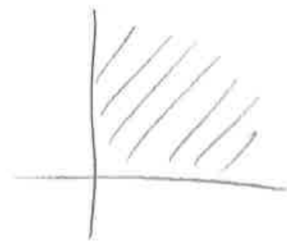
14. [10 points] Compute the Gaussian integral:

$$\int_0^\infty e^{-\pi x^2}\,dx.$$

(1pt.) $I = \int_0^\infty e^{-\pi x^2} dx$

(3pts.) $I^2 = \left(\int_0^\infty e^{-\pi x^2} dx\right)\left(\int_0^\infty e^{-\pi y^2} dy\right)$

(3pts.) $= \int_0^\infty \int_0^\infty e^{-\pi(x^2+y^2)} dx\,dy$



(1pt.) $= \int_0^{\pi/2} \int_0^\infty e^{-\pi r^2} \cdot r\,dr\,d\theta$

(1pt.) $= \frac{\pi}{2} \cdot \left(\frac{-1}{2\pi} e^{-\pi r^2}\right)\Big|_0^\infty$

(½pt.) $= \frac{1}{4}$

(½pt.) $\Rightarrow I = 1/2$