Name: ____

April 30th, 2015.
Math 2401; Sections K1, K2, K3.
Georgia Institute of Technology
FINAL EXAM

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____

Problem	Possible Score	Earned Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
Total	140	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

Angle θ ($0 \le \theta \le \pi$) between vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Vector Projection of $\mathbf{u}$ onto $\mathbf{v} \neq 0$:

$$\mathrm{Proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = |\mathbf{u}\cos\theta|\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Distance from a point S to a line L going through P and parallel to $\mathbf{v}$:

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$$

Length of a smooth curve C: $\mathbf{r}(t)$, traced exactly once as $a \le t \le b$:

$$L = \int_a^b |\mathbf{v}(t)|\,dt.$$

TNB Frame:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|};\quad \mathbf{N} = \frac{{}^{d\mathbf{T}}/_{ds}}{\kappa} = \frac{{}^{d\mathbf{T}}/_{dt}}{|{}^{d\mathbf{T}}/_{dt}|};\quad \mathbf{B} = \mathbf{T}\times\mathbf{N}.$$

Curvature:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

Tangential and Normal Components of Acceleration:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N};$$

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}(t)|;$$

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}(t)|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}.$$

Torsion:

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}.$$

Directional Derivative of f at P_0 in the direction of the unit vector $\mathbf{u}$:

$$(D_{\mathbf{u}}f)_{P_0} = (\nabla f)_{P_0}\cdot\mathbf{u}.$$

Spherical Coordinates: (ρ, ϕ, θ):

$$0 \le \phi \le \pi;\ 0 \le \theta \le 2\pi;$$

$$x = \rho\sin\phi\cos\theta;\ y = \rho\sin\phi\sin\theta;\ z = \rho\cos\phi;$$

$$\text{Jacobian: } dV \mapsto \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Green's Theorem in the Plane:

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dA;$$

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

Area with Green's Theorem:

$$\mathrm{Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Surface Differential on Parametric Surface S : $\mathbf{r}(u,v)$; $(u,v) \in R$:

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\ d(u,v)$$

Unit Normal Field on Parametric Surface S : $\mathbf{r}(u,v)$; $(u,v) \in R$:

$$\mathbf{n} = \pm\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$$

Surface Differential on Implicitly Defined (Level Surface) $f(x,y,z) = c$, over shadow region R in a coordinate plane:

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA,$$

where $\mathbf{p}$ is one of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.
Unit Normal Field on Implicitly Defined (Level Surface) $f(x,y,z) = c$, over shadow region R in a coordinate plane:

$$\mathbf{n} = \pm\frac{\nabla f}{|\nabla f|}.$$

Parametrized Sphere of radius R, centered at the origin: $0 \le \phi \le \pi$; $0 \le \theta \le 2\pi$;

$$\mathbf{r}(\phi,\theta) = R\left\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\right\rangle;$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = R^2\left\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right\rangle;$$

$$|\mathbf{r}_\phi\times\mathbf{r}_\theta| = R^2\sin\phi.$$

Stokes' Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma,$$

(with the appropriate assumptions on C, S and $\mathbf{F}$.)

Divergence Theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV,$$

(with the appropriate assumptions on S, D and $\mathbf{F}$.)

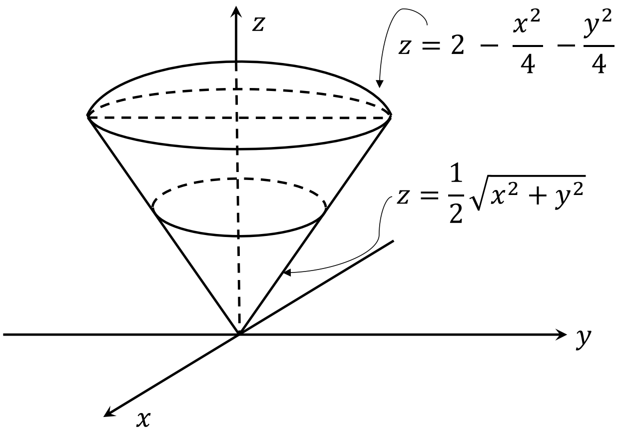
1. [10 points] Set up a triple integral in *cylindrical coordinates* that gives the volume of the "ice cream cone," the solid bounded by the cone

$$z = \frac{1}{2}\sqrt{x^2 + y^2}$$

and the paraboloid

$$z = 2 - \frac{x^2}{4} - \frac{y^2}{4}.$$

You do not need to compute the integral, just set it up!



2. [10 points] Consider the curve:

$$\vec{r}(t) = (t\sin t + \cos t)\,\vec{i} + (-t\cos t + \sin t)\,\vec{j}; \quad -\sqrt{2} \leq t \leq 0.$$

a). [3 points] Find the velocity $\vec{v}(t)$.

b). [4 points] Find the unit tangent vector $\vec{T}(t)$.

c). [3 points] Find the length of the curve.

3. [10 points] Find the points on the cone $x^2 + y^2 = z^2$ that are closest to the point $(4, 2, 0)$.

4. [10 points] Given that for a curve $\mathbf{r}(t)$:

$$\frac{d\mathbf{r}}{dt} = 3\sqrt{t+1}\,\mathbf{i} + 4e^{-t}\,\mathbf{j} + \frac{1}{t+1}\,\mathbf{k},$$

and that:

$$\mathbf{r}(0) = \langle 1, 0, 2 \rangle\,,$$

find $\mathbf{r}(t)$.

5. [10 points] Find all the critical points of $f(x, y) = xy^2 - x^2 - 2y^2$ and classify each one as either a local minimum, a local maximum, or a saddle point.

6. [10 points] Recall that the angle between two planes is defined to be the angle between their normal vectors. Consider the planes:

$$P_1 : x + y + z = -1;$$

$$P_2 : x + 2y + 3z = -4.$$

a). [3 points] Find the angle between the two planes above (give an exact answer).

b). [7 points] Find parametric equations for the line of intersection of the two planes above.

7. [10 points] Let $f(x, y)$ have continuous first order partial derivatives. Consider the points:

$$A(1,2);\;\; B(2,2);\;\; C(1,3);\;\; D(5,6).$$

Suppose that:

- the directional derivative of f at the point A in the direction of $\overrightarrow{AB}$ is equal to 2.
- the directional derivative of f at the point A in the direction of $\overrightarrow{AC}$ is equal to 4.

Use this information to find the directional derivative of f at the point A in the direction of $\overrightarrow{AD}$.

8. [10 points] Let S be the surface consisting of the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 10$, together with its "top," $x^2 + y^2 \leq 4$, $z = 10$. Let:

$$\mathbf{F}(x, y, z) = -2y\mathbf{i} + 2x\mathbf{j} + 2x^2\mathbf{k}.$$

Find the *outward flux* *of the curl* $\nabla \times \mathbf{F}$ through S.

9. [10 points] Compute the integral:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\, dx\, dy.$$

Sketch the region of integration.

10. [10 points] Compute the integral:

$$\iint_R \frac{\sin y}{y}\, dA,$$

where R is the region in the plane given by:

$$R : 1 \le y \le 3;\;\; y \le x \le 2y,$$

and sketch the region of integration.

11. [10 points] Compute the line integral

$$\int_C (2 + x^2 y)\, ds,$$

where C is the lower half of the unit circle $x^2 + y^2 = 1$ (going from $(-1, 0)$ to $(1, 0)$ along the unit circle, below the x-axis).

12. [10 points] Find the work done by the field

$$\mathbf{F}(x, y) = 5xy^3\mathbf{i} + 9x^2y^2\mathbf{j}$$

in moving a particle once counterclockwise around the curve C: the boundary of the region enclosed by the x-axis, the line $x = 1$ and the curve $y = x^3$ in the first quadrant.

13. [10 points] Find the outward flux of the field

$$\mathbf{F}(x,y,z) = \left(2x^3 + 9xy^2\right)\mathbf{i} + \left(-y^3 + \pi e^y \sin(z)\right)\mathbf{j} + \left(2z^3 + \pi e^y \cos(z)\right)\mathbf{k},$$

through the boundary of the region D:

$$D: \ 1 \le x^2 + y^2 + z^2 \le 2.$$

14. [10 points] Compute the Gaussian integral:

$$\int_0^{\infty} e^{-\pi x^2}\, dx.$$

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