Name: Solutions

April 15^{th}, 2015.
Math 2401; Sections K1, K2, K3.
Georgia Institute of Technology
Exam 4

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ____________________

Problem	Possible Score	Earned Score
0	10	10
1	18	
2	18	
3	18	
4	18	
5	18	
Total	100	

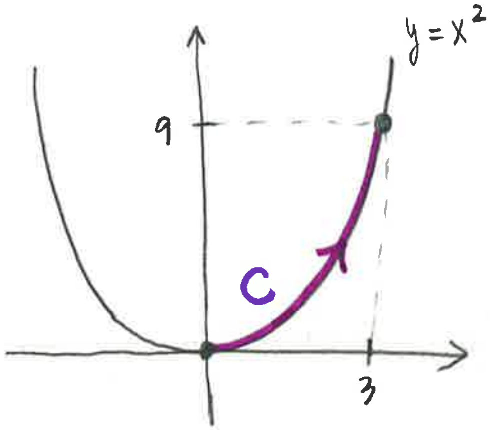
Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

1. [18 points] Find the line integral:

$$\int_C 3x\, ds,$$

where C is the portion of the parabola $y = x^2$ from $(0,0)$ to $(3,9)$.



Parametrize C: (4pts.)

$t = x$ (1)

$\vec{r}(t) = \langle t, t^2 \rangle$; (2)

$0 \le t \le 3$ (1)

Velocity: (4 pts.)

$\vec{v}(t) = \langle 1, 2t \rangle$; (2)

Speed:

$|\vec{v}(t)| = \sqrt{1+4t^2}$ (2)

(2pts.) Evaluate f on the curve: $f(x,y) = 3x$ (1/2)

$f(\vec{r}(t)) = 3t$; (3/2)

(8pts.) Compute line integral:

$$\int_C 3x\, ds = \int_0^3 f(\vec{r}(t))\, |\vec{v}(t)|\, dt$$ (2)

$$= \int_0^3 3t\sqrt{1+4t^2}\, dt$$

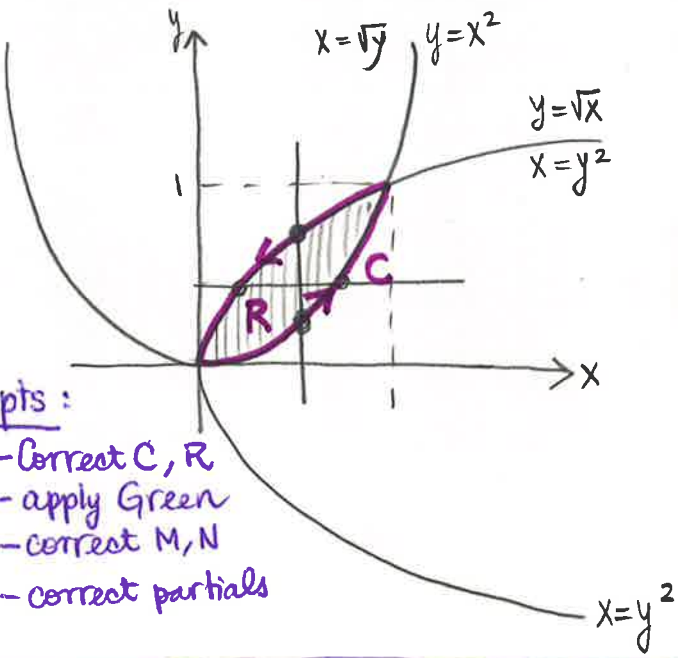
$$= \cancel{3} \cdot \frac{1}{4} \cdot \cancel{\frac{1}{2}} \cdot \cancel{\frac{2}{3}} (1+4t^2)^{3/2} \Big|_0^3$$ (4)

$$= \boxed{\frac{1}{4}\left(37^{3/2} - 1\right)}$$ (2)

2. [18 points] Find:

$$\oint_C \left(y + e^{\sqrt{x}}\right)\, dx + \left(2x + \cos y^2\right)\, dy,$$

where C is the positively oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.



9 pts:

3 - Correct C, R

2 - apply Green

2 - correct M, N

2 - correct partials

$$\oint_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$$

$$= \oint_C M\,dx + N\,dy$$

$$M = y + e^{\sqrt{x}}$$
$$N = 2x + \cos y^2$$

$$\underset{\text{(Green)}}{=} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_R (2-1)\, dA$$

$$= \text{Area}(R)$$

9 pts:

Vertical Cross-Sections: (or) Horizontal Cross-Sections:

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 1\, dy\, dx \quad — \quad 3 \quad — \quad = \int_0^1 \int_{y^2}^{\sqrt{y}} 1\, dx\, dy$$

$$= \int_0^1 y \Big|_{y=x^2}^{y=\sqrt{x}} dx \quad — \quad 2 \quad — \quad = \int_0^1 x \Big|_{x=y^2}^{x=\sqrt{y}} dy$$

$$= \int_0^1 (\sqrt{x} - x^2)\, dx \quad — \quad 1 \quad — \quad = \int_0^1 (\sqrt{y} - y^2)\, dy$$

$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_0^1 \quad — \quad 2 \quad — \quad = \left(\frac{2}{3}y^{3/2} - \frac{1}{3}y^3\right)\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} \qquad\qquad = \frac{2}{3} - \frac{1}{3}$$

$$= \boxed{\frac{1}{3}} \quad — \quad 1 \quad — \quad = \boxed{\frac{1}{3}}$$

3. [18 points] Consider the conservative field:

$$\mathbf{F}(x,y,z) = (yz)\mathbf{i} + (xz - 2y\ln(z))\mathbf{j} + \left(xy - \frac{y^2}{z}\right)\mathbf{k}.$$

a). [12 points] Find a potential function for this field.
b). [6 points] Find $\int_C \mathbf{F}\cdot dr$, where C is the curve:

$$\mathbf{r}(t) = \langle t, t^2, e^t \rangle, \ 0 \le t \le 1.$$

a). $\vec{F} = \nabla f(x,y,z)$.

1 $\frac{\partial f}{\partial x} = yz \Rightarrow f = xyz + g(y,z)$ 2

$\Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y}$ 1

$= xz - 2y\ln(z)$

1 $\Rightarrow \frac{\partial g}{\partial y} = -2y\ln(z)$ 1

$\Rightarrow g = -y^2\ln(z) + h(z)$ 2

$\Rightarrow f = xyz - y^2\ln z + h(z)$ 1

$\Rightarrow \frac{\partial f}{\partial z} = xy - \frac{y^2}{z} + h'(z)$ 1

$= xy - \frac{y^2}{z}$

$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C$ 1

$\boxed{f(x,y,z) = xyz - y^2\ln z + C}$ 1

b). Start point on C: $\vec{r}(0) = \langle 0, 0, 1\rangle$ 2

End point on C: $\vec{r}(1) = \langle 1, 1, e\rangle$ 2

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = f(1,1,e) - f(0,0,1)$ 1

$= (e-1) - (0-0)$ 1

$= \boxed{e-1}$

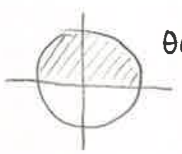
4. [18 points] Consider the following parametrized surface:

$$\mathbf{r}(\theta, z) = (5\sin(2\theta))\,\mathbf{i} + \left(10\sin^2\theta\right)\mathbf{j} + z\mathbf{k}, \quad 0 \le \theta \le \pi.$$

a). [10 points] Find an equation of the plane tangent to this surface at $P_0(0, 10, 2)$.
b). [8 points] Find a Cartesian equation for this surface and use it to sketch the surface.
Reminder: $\sin 2\theta = 2\sin\theta\cos\theta$.

(a).

- $\vec{r}_\theta = \langle 10\cos(2\theta),\ 20\sin\theta\cos\theta,\ 0 \rangle$ 2
- $\vec{r}_z = \langle 0,\ 0,\ 1 \rangle$ 2
- $\vec{r}_\theta \times \vec{r}_z = \langle 20\sin\theta\cos\theta,\ -10\cos(2\theta),\ 0 \rangle$ 2
- (θ, z) corresponding to $P_0(0,10,2)$: $\boxed{z = 2}$ 1



$\theta \in [0, \pi]$ $\quad \sin(2\theta) = 0 \Rightarrow \theta = 0$ or $\pi/2$

$\sin^2\theta = 1 \Rightarrow \boxed{\theta = \pi/2}$ 1

- Normal vector to the plane: $(\vec{r}_\theta \times \vec{r}_z)\big|_{(\pi/2, 2)} = \langle 0, 10, 0 \rangle$ 1
- Equation of plane: $10(y - 10) = 0$

$\boxed{y = 10}$ 1

(b). 1 $x = 5\sin(2\theta) = 10\sin\theta\cos\theta$

1 $y = 10\sin^2\theta$

$z = z$

$\Rightarrow x^2 + y^2 = 100\sin^2\theta\cos^2\theta + 100\sin^4\theta$ 2

$= 100\sin^2\theta\,(\cos^2\theta + \sin^2\theta)$ 1

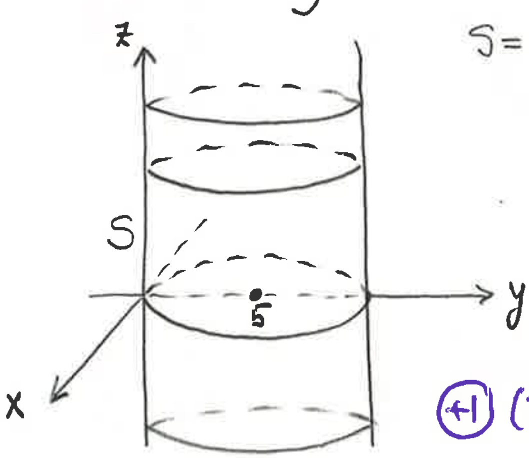
$= 100\sin^2\theta$ 1

$= 10y$ 1

$\Rightarrow x^2 + y^2 - 10y = 0$

$\Rightarrow x^2 + y^2 - 10y + 25 = 25$

1 $\Rightarrow \boxed{x^2 + (y-5)^2 = 25}$



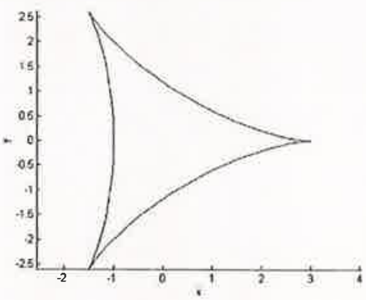
S = infinite cylinder based on circle $x^2 + (y-5)^2 = 25$, parallel to z-axis.

(+1) (Bonus) for sketch

5. [18 points] Compute the area enclosed by the deltoid curve, pictured below, and parametrized by:

$$\mathbf{r}(\theta) = \langle 2\cos\theta + \cos(2\theta), 2\sin\theta - \sin(2\theta)\rangle, \quad 0 \le \theta \le 2\pi.$$

Reminders: $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.



Green: $A = \frac{1}{2}\oint_C x\,dy - y\,dx$ 1

$x = 2\cos\theta + \cos(2\theta)$ 2

$y = 2\sin\theta - \sin(2\theta)$ 2

$dx = (-2\sin\theta - 2\sin(2\theta))\,d\theta$ 2

$dy = (2\cos\theta - 2\cos(2\theta))\,d\theta$ 2

4

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (2\cos\theta + \cos(2\theta))(2\cos\theta - 2\cos(2\theta)) + (2\sin\theta - \sin(2\theta))(+2\sin\theta + 2\sin(2\theta))\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \Big(\underbrace{4\cos^2\theta} - 4\cos\theta\cos(2\theta) + 2\cos\theta\cos(2\theta) - \underbrace{2\cos^2(2\theta)}$$

$$+ \underbrace{4\sin^2\theta}_{4} + 4\sin\theta\sin(2\theta) - 2\sin\theta\sin(2\theta) - \underbrace{2\sin^2(2\theta)}_{-2} \Big)$$

4

$$= \frac{1}{2}\int_0^{2\pi} \left(2 - 2\cos\theta\cos(2\theta) + 2\sin\theta\sin(2\theta)\right) d\theta$$

$$= \frac{1}{2}\left(4\pi - 2\int_0^{2\pi} \left(\cos\theta(1 - 2\sin^2\theta) - 2\sin^2\theta\cos\theta\right) d\theta\right)$$

$$= 2\pi - \int_0^{2\pi} \left(\cos\theta - 4\cos\theta\sin^2\theta\right) d\theta$$

$$= 2\pi - \left(\sin\theta - 4\frac{\sin^3\theta}{3}\right)\Big|_0^{2\pi}$$

1 $= \boxed{2\pi}$