Name: Solutions

February 18^{th}, 2015.
Math 2401; Sections K1, K2, K3.
Georgia Institute of Technology
Exam 2

I commit to uphold the ideals of honor and integrity by refusing to betray the trust bestowed upon me as a member of the Georgia Tech community. By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible Score	Earned Score
1	18	
2	16	
3	17	
4	16	
5	18	
6	15	
Total	100	

Remember that you must SHOW YOUR WORK to receive credit!

Good luck!

[18 pts.]

1. (a). Find the limit, or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+2y^2}.$$

Linear paths: $y = kx$ (5 pts.)

$f(x,y) = \frac{3x^2}{x^2+2y^2}$; $f(x,y)\big|_{y=kx} = \frac{3x^2}{x^2+2k^2x^2} = \frac{3}{1+2k^2} \xrightarrow[x\to 0]{} \frac{3}{1+2k^2}$

$\Rightarrow$ Limit DNE by the Two-Path Test. (4 pts.)

(b). Find the limit, or show that it does not exist:

$$\lim_{(x,y)\to(4,1)} \frac{\sqrt{x}-2\sqrt{y}}{x-4y}.$$

$\frac{\sqrt{x}-2\sqrt{y}}{x-4y} = \frac{\cancel{\sqrt{x}-2\sqrt{y}}}{\cancel{(\sqrt{x}-2\sqrt{y})}(\sqrt{x}+2\sqrt{y})} = \frac{1}{\sqrt{x}+2\sqrt{y}}$ (5 pts.)

$\lim_{(x,y)\to(4,1)} \frac{1}{\sqrt{x}+2\sqrt{y}} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$ (4 pts.)

[16pts.]

2. Find equations for the tangent plane to the surface given by:

$$\sin(xyz) = x + 2y + 3z$$

at the point $(2, -1, 0)$.

(2 pts.) $f(x,y,z) = \sin(xyz) - x - 2y - 3z$ $\quad$ (or $f(x,y,z) = x + 2y + 3z - \sin(xyz)$)

Level surface : $f(x,y,z) = 0$

(9 pts.) $\nabla f = \langle yz\cos(xyz) - 1,\ \ xz\cos(xyz) - 2,\ \ xy\cos(xyz) - 3 \rangle$

(2 pts.) $\nabla f \big|_{(2,-1,0)} = \langle 0-1,\ 0-2,\ -2-3 \rangle = \langle -1, -2, -5 \rangle$

(1 pt.) $\nabla f \big|_{(2,-1,0)}$ = normal vector to tangent plane

Tangent plane : $-1(x-2) - 2(y+1) - 5(z-0) = 0$

(2 pts.)

$$-x + 2 - 2y - 2 - 5z = 0$$

$$\boxed{x + 2y + 5z = 0}$$

[17pts.]

3. Find the directions in which the directional derivative of $f(x,y) = x^2 + \sin(xy)$ at the point $(1,0)$ is equal to 1.

(6pts.) $\nabla f = \langle 2x + y\cos(xy),\ x\cos(xy) \rangle$

(2pts.) $\nabla f \big|_{(1,0)} = \langle 2, 1 \rangle$

(1pt.) $D_{\vec{u}} f(1,0) = 1$; $\vec{u} = \langle a, b \rangle$ $\Rightarrow$ $\langle 2,1 \rangle \cdot \langle a, b \rangle = 2a + b = 1$

(2pts.) Set-up $\begin{cases} 2a + b = 1 \\ a^2 + b^2 = 1 \end{cases}$ $\quad \begin{cases} b = 1 - 2a \\ a^2 + (1-2a)^2 = 1 \end{cases}$ $\quad \begin{cases} b = 1 - 2a \\ a^2 + \cancel{1} - 4a + 4a^2 = \cancel{1} \end{cases}$

(4pts.) – solving system

$5a^2 - 4a = 0$

$a(5a - 4) = 0 \Rightarrow a = 0$ or $a = \frac{4}{5}$

$a = 0 \Rightarrow b = 1$

$a = \frac{4}{5} \Rightarrow b = 1 - \frac{8}{5} = -\frac{3}{5}$

(2pts.) $\boxed{\vec{u} = \langle 0, 1 \rangle \text{ or } \vec{u} = \langle 4/5, -3/5 \rangle}$

[16pts.]

4. Suppose $f(x, y, z)$ has continuous first-order partial derivatives and:

$$f_x(2,1,-1) = 3; \;\; f_x(4,3,2) = -2;$$

$$f_y(2,1,-1) = -5; \;\; f_y(4,3,2) = -2;$$

$$f_z(2,1,-1) = 7; \;\; f_z(4,3,2) = -1.$$

If g is given by:

$$g(t) = f\left(2t^2, t^3, -\frac{1}{t^2}\right),$$

find $g'(1)$.

(3pts.)
$$\left.\begin{aligned} x(t) &= 2t^2 \\ y(t) &= t^3 \\ z(t) &= -\frac{1}{t^2} \end{aligned}\right\} \Rightarrow g(t) = f(x(t), y(t), z(t)) \quad \text{(2pts.)}$$

$$\Rightarrow g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \quad \text{(3pts.)}$$

$$= (f_x)(4t) + (f_y)(3t^2) + (f_z)\left(\frac{2}{t^3}\right) \quad \text{(3pts.)}$$

$$t = 1 \Rightarrow (x, y, z) = (2, 1, -1) \quad \text{(3pts.)}$$

$$\Rightarrow g'(1) = f_x(2,1,-1)\cdot 4 + f_y(2,1,-1)\cdot 3 + f_z(2,1,-1)\cdot 2 \quad \text{(1pt.)}$$

$$= 3\cdot 4 - 5\cdot 3 + 7\cdot 2$$

$$= 12 - 15 + 14$$

$$= \boxed{11}. \quad \text{(1pt.)}$$

[18pts.]

5. Find all the critical points of the function:

$$f(x, y) = x^4 + 4xy + xy^2$$

and classify each one as a local minimum, a local maximum, or a saddle point.

(1pt.) $f_x = 4x^3 + 4y + y^2$

(1pt.) $f_y = 4x + 2xy = 2x(2+y)$

(1pt.) $f_{xx} = 12x^2$

(1pt.) $f_{yy} = 2x$

$f_{xy} = 4 + 2y$ (1pt.)

(1pt.) $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$ $\quad \begin{cases} 4x^3 + 4y + y^2 = 0 & (1) \\ 2x(2+y) = 0 & (2) \end{cases}$

From (2): either $x = 0$ or $y = -2$

(5pts.) Solving system

$x = 0 \Rightarrow$ (1) becomes: $4y + y^2 = 0$

$y(4+y) = 0$ $\Rightarrow (0,0)$ & $(0,-4)$

$y = 0$ or $y = -4$

$y = -2 \Rightarrow$ (1) becomes: $4x^3 - 8 + 4 = 0$

$4x^3 - 4 = 0$

$4(x^3 - 1) = 0$ $\Rightarrow (1,-2)$

$x = 1$

(1pt.) Critical Points : $(0,0)$; $(0,-4)$; $(1,-2)$.

$$\Delta_f(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 24x^3 - (4+2y)^2$$

(2pts.) $\Delta_f(0,0) < 0 \Rightarrow$ saddle point

(2pts.) $\Delta_f(0,-4) < 0 \Rightarrow$ saddle point

(2pts.) $\left.\begin{array}{l} \Delta_f(1,-2) = 24 - 0 > 0 \\ f_{xx}(1,-2) > 0 \end{array}\right\} \Rightarrow$ local minimum

[15pts.]

6. Find the minimum and maximum of the function:

$$f(x, y) = e^{-xy}$$

on the region described by:

$$x^2 + 4y^2 \leq 1.$$

Critical Points : $f_x = -ye^{-xy}$ $\quad f_y = -xe^{-xy}$ $\quad \begin{cases} -ye^{-xy} = 0 \\ -xe^{-xy} = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$

The only critical point is $(0,0)$, and it lies within the region, so we evaluate f there: $f(0,0) = e^0 = 1$.

(x,y)	$f(x,y)$
$(0,0)$	1

Boundary : Find extreme values of $f(x,y)$ subject to the restriction $x^2+4y^2=1$.

Solution (A): Lagrange Multipliers : $g(x,y) = x^2+4y^2$;

$\nabla f = \langle -ye^{-xy}, -xe^{-xy} \rangle$; $\nabla g = \langle 2x, 8y \rangle$

Solve : $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{cases}$ $\quad \begin{cases} -ye^{-xy} = \lambda \cdot 2x \\ -xe^{-xy} = \lambda \cdot 8y \\ x^2+4y^2 = 1 \end{cases}$ $\quad \begin{cases} -ye^{-xy} = 2\lambda x & (1) \\ -xe^{-xy} = 8\lambda y & (2) \\ x^2+4y^2 = 1 & (3) \end{cases}$

Solution 1 : Multiply (1) by x and (2) by y :

$$\left.\begin{array}{l} -xye^{-xy} = 2\lambda x^2 \\ -xye^{-xy} = 8\lambda y^2 \end{array}\right\} \Rightarrow 2\lambda x^2 = 8\lambda y^2 \Rightarrow 2\lambda(x^2-4y^2) = 0$$

$\Rightarrow \lambda = 0$ or $x^2 - 4y^2 = 0$

- $\lambda = 0 \Rightarrow$ from (1), (2) : $x = y = 0$
  Replace $x=y=0$ in (3) : $0 = 1$ false! $\Rightarrow \lambda \neq 0$

- $x^2 - 4y^2 = 0$
  $x^2 + 4y^2 = 1$

$\oplus$ $2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow 4y^2 = \frac{1}{2} \Rightarrow y^2 = \frac{1}{8}$

$\Rightarrow \boxed{(x,y) \in \left\{ \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right), \left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) \right\}}$

Solution 2 : Multiply (1) by y and (2) by x :

$$\left.\begin{aligned} -y^2e^{-xy} &= 2\lambda xy \\ -x^2e^{-xy} &= 8\lambda xy \end{aligned}\right\} \Rightarrow -x^2e^{-xy} = 4\left(-y^2e^{-xy}\right)$$

$$\Rightarrow \left(4y^2 - x^2\right)e^{-xy} = 0$$

$$\Rightarrow 4y^2 - x^2 = 0$$

$$4y^2 - x^2 = 0$$

$$(2y - x)(2y + x) = 0$$

$$x = 2y \quad \text{or} \quad x = -2y$$

$x = 2y \Rightarrow$ (3) becomes : $4y^2 + 4y^2 = 1 \Rightarrow y^2 = \frac{1}{8} \Rightarrow y = \pm\frac{1}{2\sqrt{2}} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$x = -2y \Rightarrow$ (3) becomes : $4y^2 + 4y^2 = 1 \Rightarrow y^2 = \frac{1}{8} \Rightarrow y = \pm\frac{1}{2\sqrt{2}} \Rightarrow x = \mp\frac{1}{\sqrt{2}}$

$$\boxed{(x,y) \in \left\{\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right), \left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right)\right\}}$$

Evaluate f at $\underbrace{\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right)}_{\hookrightarrow\, xy = 1/4}$, $\underbrace{\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right)}_{\hookrightarrow\, xy = -1/4}$

(x,y)	$f(x,y)$	
$(0,0)$	$1 = e^0$	
$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right)$	$e^{-1/4}$	$\leftsquigarrow$ min
$\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right)$	$e^{1/4}$	$\leftsquigarrow$ max

Boundary Solution (B): Parametrize the boundary (ellipse): $x^2+4y^2=1$

$$x^2+\frac{y^2}{(1/2)^2}=1$$

$$\begin{cases} x=\cos t \\ y=\frac{1}{2}\sin t \end{cases} \qquad 0\le t\le 2\pi$$

$$g(t)=f\left(\cos t, \tfrac{1}{2}\sin t\right)=e^{-\frac{1}{2}\cos t\sin t}=e^{-\frac{1}{4}\sin(2t)}, \quad t\in[0,2\pi]$$

$$g'(t)=-\frac{1}{2}\cos(2t)\,e^{-1/4\sin(2t)}$$

$$g'(t)=0 \Rightarrow \left.\begin{matrix}\cos(2t)=0 \\ 0\le t\le 2\pi\end{matrix}\right\} \Rightarrow 2t=\pi/2 \text{ or } 3\pi/2, \; 5\pi/2 \text{ or } 7\pi/2 \Rightarrow t=\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$$

Evaluate g at the critical points $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ and at the endpoints $0, 2\pi$:

$$g(5\pi/4)=g(\pi/4)=e^{-1/4\sin(\pi/2)}=e^{-1/4} \quad \text{min}$$

$$g(7\pi/4)=g(3\pi/4)=e^{-1/4\sin(3\pi/2)}=e^{1/4} \quad \text{max}$$

$$g(0)=g(2\pi)=e^{-1/4\cdot 0}=1$$

$$f(0,0)=1$$

Find c.pt. $(0,0)$: 4pts.

Evaluate $f(0,0)$: 1 pt.

Boundary: Finding the extreme values $e^{1/4}, e^{-1/4}$ (any method): 9 pts.

Find absolute min/max of f over the region: 1 pt.