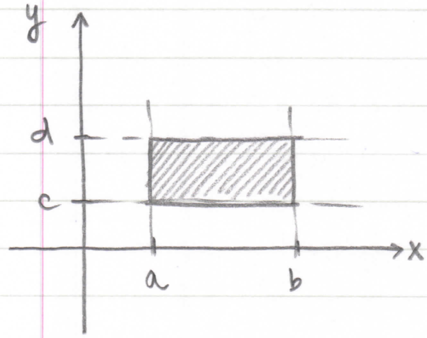
Chapter 15 Review

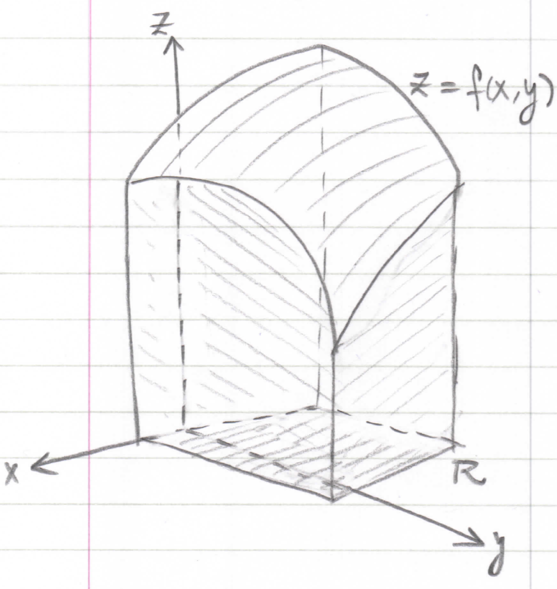
15.1 Double Integrals over Rectangles



- Rectangle R: $a \le x \le b;\ c \le y \le d$.
- Integral of a continuous function $f(x,y)$ over R:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

(Fubini's Theorem).

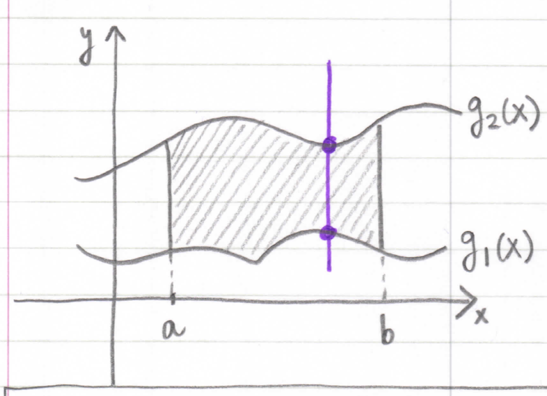


- Volume of solid bounded above by surface $z = f(x,y)$ and below by a rectangle R in the xy-plane:

$$\boxed{V = \iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx}$$

15.2 Double Integrals over "General" Regions:

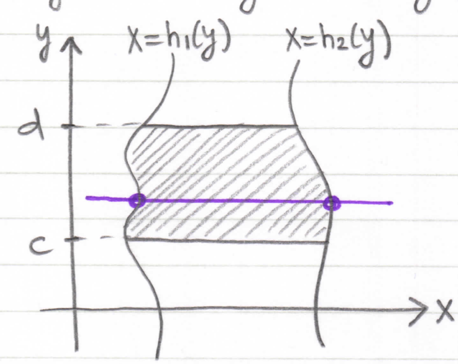
Region R: $a \le x \le b,\ g_1(x) \le y \le g_2(x)$



$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

(Vertical Cross-Sections)

Region R: $h_1(y) \le x \le h_2(y),\ c \le y \le d$



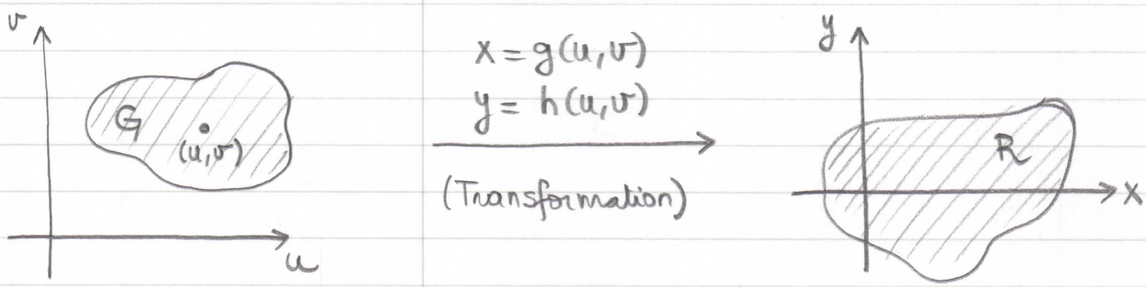
$$\boxed{\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

(Horizontal Cross-Sections).

15.3 Area by Double Integration:

- Area of a closed bounded region R in the plane: $\boxed{\iint_R dA}$
- Average value of $f(x,y)$ over R: $\boxed{\dfrac{1}{\text{area}(R)} \iint_R f(x,y)\,dA}$

15.8 Substitution in Double Integrals:



<u>Jacobian</u> of transformation $(x = g(u,v),\ y = h(u,v))$:

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}$$

If g, h, f have continuous partial derivatives and $J(u,v)$ is <u>never</u> 0:

$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v), h(u,v))\ |J(u,v)|\,du\,dv}$$

15.5 Triple Integrals:

- Volume of a closed bounded region D in space: $\boxed{V = \iiint_D dV}$
- Average value of $f(x,y,z)$ over D: $\boxed{\dfrac{1}{\text{volume}(D)} \iiint_D f(x,y,z)\,dV}$

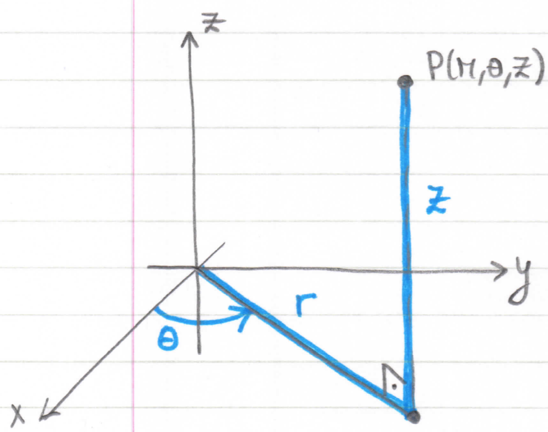
15.4 Double Integrals in Polar Coordinates :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad J(r,\theta) = r \qquad \boxed{dx\,dy} \mapsto \boxed{r\,dr\,d\theta}$$

15.7 Cylindrical & Spherical Coordinates – Triple Integrals

Cylindrical Coordinates : (r, θ, z)

$P(r,\theta,z) = P(x,y,z)$
- (r,θ) = polar coordinates of (x,y)
- z = z-coordinate

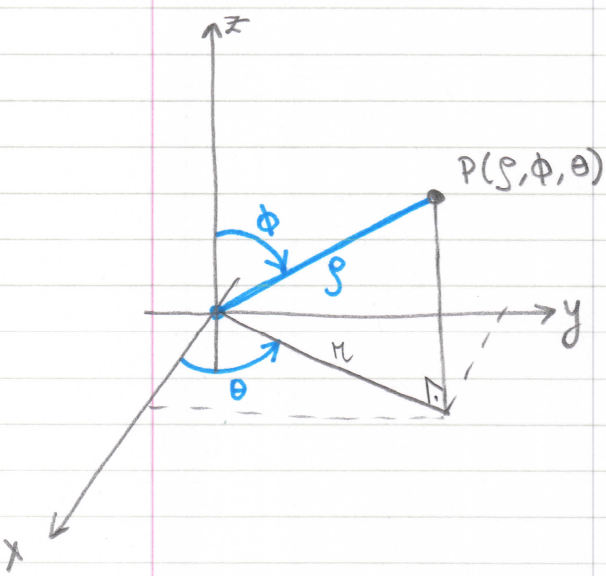


$$\boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad r^2 = x^2+y^2}$$

$$\boxed{dx\,dy\,dz} \mapsto \boxed{r\,dz\,dr\,d\theta}$$

Spherical Coordinates : (ρ, ϕ, θ)

$P(\rho,\phi,\theta)$ $\left(\rho \geq 0,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi\right)$
- ρ = distance from P to origin
- ϕ = angle between $\vec{OP}$ and the positive z-axis
- θ = angle from cylindrical coordinates



$$\boxed{\begin{cases} r = \rho\sin\phi \\ z = \rho\cos\phi \\ x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \end{cases} \quad \rho^2 = x^2+y^2+z^2 = r^2+z^2}$$

$$\boxed{dx\,dy\,dz} \downarrow \boxed{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$